Discussion and Writing

83. Write a brief paragraph that explains how to quickly compute the trigonometric functions of 30°, 45°, and 60°.

84. Explain how you would measure the width of the Grand Canyon from a point on its ridge.

85. Explain how you would measure the height of a TV tower that is on the roof of a tall building.

7.4 Trigonometric Functions of General Angles

**OBJECTIVES**

1. Find the Exact Values of the Trigonometric Functions for General Angles (p. 540)
2. Use Coterminal Angles to Find the Exact Value of a Trigonometric Function (p. 542)
3. Determine the Signs of the Trigonometric Functions of an Angle in a Given Quadrant (p. 544)
4. Find the Reference Angle of a General Angle (p. 545)
5. Use a Reference Angle to Find the Exact Value of a Trigonometric Function (p. 546)
6. Find the Exact Values of Trigonometric Functions of an Angle, Given Information about the Functions (p. 547)

**DEFINITION**

Let \( \theta \) be any angle in standard position, and let \((a, b)\) denote the coordinates of any point, except the origin \((0, 0)\), on the terminal side of \( \theta \). If \( r = \sqrt{a^2 + b^2} \) denotes the distance from \((0, 0)\) to \((a, b)\), then the six trigonometric functions of \( \theta \) are defined as the ratios

\[
\sin \theta = \frac{b}{r} \quad \cos \theta = \frac{a}{r} \quad \tan \theta = \frac{b}{a}
\]

\[
\csc \theta = \frac{r}{b} \quad \sec \theta = \frac{r}{a} \quad \cot \theta = \frac{a}{b}
\]

provided no denominator equals 0. If a denominator equals 0, that trigonometric function of the angle \( \theta \) is not defined.

Notice in the preceding definitions that if \( a = 0 \), that is, if the point \((a, b)\) is on the \( y \)-axis, then the tangent function and the secant function are undefined. Also, if \( b = 0 \), that is, if the point \((a, b)\) is on the \( x \)-axis, then the cosecant function and the cotangent function are undefined.

By constructing similar triangles, you should be convinced that the values of the six trigonometric functions of an angle \( \theta \) do not depend on the selection of the point \((a, b)\) on the terminal side of \( \theta \), but rather depend only on the angle \( \theta \) itself. See Figure 38 for an illustration of this when \( \theta \) lies in quadrant II. Since the triangles are similar, the ratio \( \frac{b}{r} \) equals the ratio \( \frac{b'}{r'} \), which equals \( \sin \theta \). Also, the ratio \( \frac{|a|}{r} \) equals the ratio \( \frac{|a'|}{r'} \), so \( \frac{a}{r} = \frac{a'}{r'} \), which equals \( \cos \theta \). And so on.
Also, observe that if $\theta$ is acute these definitions reduce to the right triangle definitions given in Section 7.2, as illustrated in Figure 39.

Finally, from the definition of the six trigonometric functions of a general angle, we see that the Quotient and Reciprocal Identities hold. Also, using $r^2 = a^2 + b^2$ and dividing each side by $r^2$, we can derive the Pythagorean Identities for general angles.

### Example 1

**Finding the Exact Values of the Six Trigonometric Functions of $\theta$, Given a Point on the Terminal Side**

Find the exact value of each of the six trigonometric functions of a positive angle $\theta$ if $(4, -3)$ is a point on its terminal side.

Figure 40 illustrates the situation. For the point $(a, b) = (4, -3)$, we have $a = 4$ and $b = -3$. Then $r = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$, so that

\[
\sin \theta = \frac{b}{r} = \frac{-3}{5} \quad \cos \theta = \frac{a}{r} = \frac{4}{5} \quad \tan \theta = \frac{b}{a} = \frac{-3}{4}
\]

\[
\csc \theta = \frac{r}{b} = \frac{-5}{3} \quad \sec \theta = \frac{r}{a} = \frac{5}{4} \quad \cot \theta = \frac{a}{b} = \frac{-4}{3}
\]

In the next example, we find the exact value of each of the six trigonometric functions at the quadrantal angles $0, \frac{\pi}{2}, \pi,$ and $\frac{3\pi}{2}$.

### Example 2

**Finding the Exact Values of the Six Trigonometric Functions of Quadrantal Angles**

Find the exact values of each of the six trigonometric functions of

(a) $\theta = 0 = 0^\circ$  
(b) $\theta = \frac{\pi}{2} = 90^\circ$  
(c) $\theta = \pi = 180^\circ$  
(d) $\theta = \frac{3\pi}{2} = 270^\circ$

(a) The point $P = (1, 0)$ is on the terminal side of $\theta = 0 = 0^\circ$ and is a distance of 1 unit from the origin. See Figure 41. Then

\[
\sin 0 = \sin 0^\circ = 0 \quad \cos 0 = \cos 0^\circ = 1
\]

\[
\tan 0 = \tan 0^\circ = 0 \quad \sec 0 = \sec 0^\circ = \frac{1}{1} = 1
\]

Since the $y$-coordinate of $P$ is 0, csc 0 and cot 0 are not defined.

(b) The point $P = (0, 1)$ is on the terminal side of $\theta = \frac{\pi}{2} = 90^\circ$ and is a distance of 1 unit from the origin. See Figure 42. Then

\[
\sin \frac{\pi}{2} = \sin 90^\circ = 1 \quad \cos \frac{\pi}{2} = \cos 90^\circ = 0
\]

\[
\csc \frac{\pi}{2} = \csc 90^\circ = \frac{1}{1} = 1 \quad \cot \frac{\pi}{2} = \cot 90^\circ = 0
\]

Since the $x$-coordinate of $P$ is 0, $\tan \frac{\pi}{2}$ and $\sec \frac{\pi}{2}$ are not defined.
(c) The point \( P = (-1, 0) \) is on the terminal side of \( \theta = \pi = 180^\circ \) and is a distance of 1 unit from the origin. See Figure 43. Then
\[
\sin \pi = \sin 180^\circ = 0, \quad \cos \pi = \cos 180^\circ = -1, \\
\tan \pi = \tan 180^\circ = 0, \quad \sec \pi = \sec 180^\circ = -1.
\]

Since the y-coordinate of \( P \) is 0, \( \csc \pi \) and \( \cot \pi \) are not defined.

(d) The point \( P = (0, -1) \) is on the terminal side of \( \theta = \frac{3\pi}{2} = 270^\circ \) and is a distance of 1 unit from the origin. See Figure 44. Then
\[
\sin \frac{3\pi}{2} = \sin 270^\circ = -1, \quad \cos \frac{3\pi}{2} = \cos 270^\circ = 0, \\
\csc \frac{3\pi}{2} = \csc 270^\circ = 0, \quad \cot \frac{3\pi}{2} = \cot 270^\circ = 0.
\]

Since the x-coordinate of \( P \) is 0, \( \tan \frac{3\pi}{2} \) and \( \sec \frac{3\pi}{2} \) are not defined.

Table 4 summarizes the values of the trigonometric functions found in Example 2.

<table>
<thead>
<tr>
<th>( \theta ) (Radians)</th>
<th>( \theta ) (Degrees)</th>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
<th>( \csc \theta )</th>
<th>( \sec \theta )</th>
<th>( \cot \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0°</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>Not defined</td>
<td>1</td>
<td>Not defined</td>
</tr>
<tr>
<td>( \frac{\pi}{2} )</td>
<td>90°</td>
<td>1</td>
<td>0</td>
<td>Not defined</td>
<td>1</td>
<td>Not defined</td>
<td>0</td>
</tr>
<tr>
<td>( \pi )</td>
<td>180°</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>Not defined</td>
<td>-1</td>
<td>Not defined</td>
</tr>
<tr>
<td>( \frac{3\pi}{2} )</td>
<td>270°</td>
<td>-1</td>
<td>0</td>
<td>Not defined</td>
<td>-1</td>
<td>Not defined</td>
<td>0</td>
</tr>
</tbody>
</table>

There is no need to memorize Table 4. To find the value of a trigonometric function of a quadrantal angle, draw the angle and apply the definition, as we did in Example 2.

2 Use Coterminal Angles to Find the Exact Value of a Trigonometric Function

**Definition**

Two angles in standard position are said to be **coterminal** if they have the same terminal side.

See Figure 45.

For example, the angles 60° and 420° are coterminal, as are the angles -40° and 320°.
In general, if $\theta$ is an angle measured in degrees, then $\theta + 360^\circ k$, where $k$ is any integer, is an angle coterminal with $\theta$. If $\theta$ is measured in radians, then $\theta + 2\pi k$, where $k$ is any integer, is an angle coterminal with $\theta$.

Because coterminal angles have the same terminal side, it follows that the values of the trigonometric functions of coterminal angles are equal. We use this fact in the next example.

**EXAMPLE 3**

Using a Coterminel Angle to Find the Exact Value of a Trigonometric Function

Find the exact value of each of the following:

(a) $\sin 390^\circ$ (b) $\cos 420^\circ$ (c) $\tan \frac{9\pi}{4}$ (d) $\sec \left(-\frac{7\pi}{4}\right)$ (e) $\csc(-270^\circ)$

**Solution**

(a) It is best to sketch the angle first. See Figure 46. The angle $390^\circ$ is coterminal with $30^\circ$.

\[
\sin 390^\circ = \sin (30^\circ + 360^\circ) = \sin 30^\circ = \frac{1}{2}
\]

(b) See Figure 47. The angle $420^\circ$ is coterminal with $60^\circ$.

\[
\cos 420^\circ = \cos (60^\circ + 360^\circ) = \cos 60^\circ = \frac{1}{2}
\]

(c) See Figure 48. The angle $\frac{9\pi}{4}$ is coterminal with $\frac{\pi}{4}$.

\[
\tan \frac{9\pi}{4} = \tan \left(\frac{\pi}{4} + 2\pi\right) = \tan \frac{\pi}{4} = 1
\]

(d) See Figure 49. The angle $-\frac{7\pi}{4}$ is coterminal with $\frac{\pi}{4}$.

\[
\sec \left(-\frac{7\pi}{4}\right) = \sec \left(\frac{\pi}{4} + 2\pi(-1)\right) = \sec \frac{\pi}{4} = \sqrt{2}
\]

(e) See Figure 50. The angle $-270^\circ$ is coterminal with $90^\circ$.

\[
\csc(-270^\circ) = \csc(90^\circ + 360^\circ(-1)) = \csc 90^\circ = 1
\]

As Example 3 illustrates, the value of a trigonometric function of any angle is equal to the value of the same trigonometric function of an angle $\theta$ coterminal to it, where $0^\circ \leq \theta < 360^\circ$ (or $0 \leq \theta < 2\pi$). Because the angles $\theta$ and $\theta + 360^\circ k$ (or $\theta + 2\pi k$), where $k$ is any integer, are coterminal, and because the values of the trigonometric functions are equal for coterminal angles, it follows that

\[
\begin{align*}
\theta \text{ degrees} & \quad \theta \text{ radians} \\
\sin(\theta + 360^\circ k) & = \sin \theta & \sin(\theta + 2\pi k) & = \sin \theta \\
\cos(\theta + 360^\circ k) & = \cos \theta & \cos(\theta + 2\pi k) & = \cos \theta \\
\tan(\theta + 360^\circ k) & = \tan \theta & \tan(\theta + 2\pi k) & = \tan \theta \\
\csc(\theta + 360^\circ k) & = \csc \theta & \csc(\theta + 2\pi k) & = \csc \theta \\
\sec(\theta + 360^\circ k) & = \sec \theta & \sec(\theta + 2\pi k) & = \sec \theta \\
\cot(\theta + 360^\circ k) & = \cot \theta & \cot(\theta + 2\pi k) & = \cot \theta
\end{align*}
\]

where $k$ is any integer.
These formulas show that the values of the trigonometric functions repeat themselves every $360^\circ$ (or $2\pi$ radians).

Now Work Problem 21

3 Determine the Signs of the Trigonometric Functions of an Angle in a Given Quadrant

If $\theta$ is not a quadrantal angle, then it will lie in a particular quadrant. In such a case, the signs of the $x$-coordinate and $y$-coordinate of a point $(a, b)$ on the terminal side of $\theta$ are known. Because $r = \sqrt{a^2 + b^2} > 0$, it follows that the signs of the trigonometric functions of an angle $\theta$ can be found if we know in which quadrant $\theta$ lies.

For example, if $\theta$ lies in quadrant II, as shown in Figure 51, then a point $(a, b)$ on the terminal side of $\theta$ has a negative $x$-coordinate and a positive $y$-coordinate. Then,

$$\sin \theta = \frac{b}{r} > 0 \quad \cos \theta = \frac{a}{r} < 0 \quad \tan \theta = \frac{b}{a} < 0$$

$$\csc \theta = \frac{r}{b} > 0 \quad \sec \theta = \frac{r}{a} < 0 \quad \cot \theta = \frac{a}{b} < 0$$

Table 5 lists the signs of the six trigonometric functions for each quadrant. Figure 52 provides two illustrations.

<table>
<thead>
<tr>
<th>Quadrant of $\theta$</th>
<th>$\sin \theta$, $\csc \theta$</th>
<th>$\cos \theta$, $\sec \theta$</th>
<th>$\tan \theta$, $\cot \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Positive</td>
<td>Positive</td>
<td>Positive</td>
</tr>
<tr>
<td>II</td>
<td>Positive</td>
<td>Negative</td>
<td>Negative</td>
</tr>
<tr>
<td>III</td>
<td>Negative</td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>IV</td>
<td>Negative</td>
<td>Positive</td>
<td>Negative</td>
</tr>
</tbody>
</table>

---

**EXAMPLE 4** Finding the Quadrant in Which an Angle Lies

If $\sin \theta < 0$ and $\cos \theta < 0$, name the quadrant in which the angle $\theta$ lies.

**Solution** If $\sin \theta < 0$, then $\theta$ lies in quadrant III or IV. If $\cos \theta < 0$, then $\theta$ lies in quadrant II or III. Therefore, $\theta$ lies in quadrant III.

Now Work Problem 33
4 Find the Reference Angle of a General Angle

Once we know in which quadrant an angle lies, we know the sign of each trigonometric function of this angle. This information, along with the reference angle will allow us to evaluate the trigonometric functions of such an angle.

**DEFINITION**

Let \( \theta \) denote an angle that lies in a quadrant. The acute angle formed by the terminal side of \( \theta \) and the \( x \)-axis is called the **reference angle** for \( \theta \).

Figure 53 illustrates the reference angle for some general angles \( \theta \). Note that a reference angle is always an acute angle. That is, a reference angle has a measure between 0° and 90°.

![Figure 53](image1)

Although formulas can be given for calculating reference angles, usually it is easier to find the reference angle for a given angle by making a quick sketch of the angle.

**EXAMPLE 5**

**Finding Reference Angles**

Find the reference angle for each of the following angles:

(a) \( 150^\circ \)  
(b) \( -45^\circ \)  
(c) \( \frac{9\pi}{4} \)  
(d) \( -\frac{5\pi}{6} \)

**Solution**

(a) Refer to Figure 54. The reference angle for \( 150^\circ \) is \( 30^\circ \).

![Figure 54](image2)

(b) Refer to Figure 55. The reference angle for \( -45^\circ \) is \( 45^\circ \).

![Figure 55](image3)

(c) Refer to Figure 56. The reference angle for \( \frac{9\pi}{4} \) is \( \frac{\pi}{4} \).

![Figure 56](image4)

(d) Refer to Figure 57. The reference angle for \( -\frac{5\pi}{6} \) is \( \frac{\pi}{6} \).

![Figure 57](image5)
5 Use a Reference Angle to Find the Exact Value of a Trigonometric Function

The advantage of using reference angles is that, except for the correct sign, the values of the trigonometric functions of a general angle \( \theta \) equal the values of the trigonometric functions of its reference angle.

**Reference Angles**

If \( \theta \) is an angle that lies in a quadrant and if \( \alpha \) is its reference angle, then

\[
\begin{align*}
\sin \theta &= \pm \sin \alpha & \cos \theta &= \pm \cos \alpha & \tan \theta &= \pm \tan \alpha \\
\csc \theta &= \pm \csc \alpha & \sec \theta &= \pm \sec \alpha & \cot \theta &= \pm \cot \alpha
\end{align*}
\]  

(2)

where the + or - sign depends on the quadrant in which \( \theta \) lies.

For example, suppose that \( \theta \) lies in quadrant II and \( \alpha \) is its reference angle. See Figure 58. If \((a, b)\) is a point on the terminal side of \( \theta \) and if \( r = \sqrt{a^2 + b^2} \), we have

\[
\begin{align*}
\sin \theta &= \frac{b}{r} = \sin \alpha & \cos \theta &= \frac{a}{r} = -\frac{|a|}{r} = -\cos \alpha \\
\end{align*}
\]

and so on.

The next example illustrates how the theorem on reference angles is used.

**EXAMPLE 6** Using the Reference Angle to Find the Exact Value of a Trigonometric Function

Find the exact value of each of the following trigonometric functions using reference angles.

(a) \( \sin 135^\circ \)  (b) \( \cos 600^\circ \)  (c) \( \cos \frac{17\pi}{6} \)  (d) \( \tan \left(-\frac{\pi}{3}\right) \)

**Solution** (a) Refer to Figure 59. The reference angle for \( 135^\circ \) is \( 45^\circ \) and \( \sin 45^\circ = \frac{\sqrt{2}}{2} \). The angle \( 135^\circ \) is in quadrant II, where the sine function is positive, so

\[
\sin 135^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2}
\]

(b) Refer to Figure 60. The reference angle for \( 600^\circ \) is \( 60^\circ \) and \( \cos 60^\circ = \frac{1}{2} \). The angle \( 600^\circ \) is in quadrant III, where the cosine function is negative, so

\[
\cos 600^\circ = -\cos 60^\circ = -\frac{1}{2}
\]

(c) Refer to Figure 61. The reference angle for \( \frac{17\pi}{6} \) is \( \frac{\pi}{6} \) and \( \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \). The angle \( \frac{17\pi}{6} \) is in quadrant II, where the cosine function is negative, so

\[
\cos \frac{17\pi}{6} = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}
\]
(d) Refer to Figure 62. The reference angle for \(-\frac{\pi}{3}\) is \(\frac{\pi}{3}\) and \(\tan\left(\frac{\pi}{3}\right) = \sqrt{3}\). The angle \(-\frac{\pi}{3}\) is in quadrant IV, where the tangent function is negative, so

\[
\tan\left(-\frac{\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}
\]

### Finding the Values of the Trigonometric Functions of a General Angle

If the angle \(\theta\) is a quadrantal angle, draw the angle, pick a point on its terminal side, and apply the definition of the trigonometric functions.

If the angle \(\theta\) lies in a quadrant:

1. Find the reference angle \(\alpha\) of \(\theta\).
2. Find the value of the trigonometric function at \(\alpha\).
3. Adjust the sign (+ or −) according to the sign of the trigonometric function in the quadrant where \(\theta\) lies.

---

**EXAMPLE 7**

Finding the Exact Values of Trigonometric Functions

Given that \(\cos \theta = -\frac{2}{3}\) and \(\frac{\pi}{2} < \theta < \pi\), find the exact value of each of the remaining trigonometric functions.

**Solution**

The angle \(\theta\) lies in quadrant II, so we know that \(\sin \theta\) and \(\csc \theta\) are positive and the other four trigonometric functions are negative. If \(\alpha\) is the reference angle for \(\theta\), then \(\cos \alpha = \frac{2}{3}\) and \(\sin \alpha = \frac{\sqrt{5}}{3}\). The values of the remaining trigonometric functions of the reference angle \(\alpha\) can be found by drawing the appropriate triangle and using the Pythagorean Theorem. We use Figure 63 to obtain

\[
\sin \alpha = \frac{\sqrt{5}}{3} \quad \cos \alpha = \frac{2}{3} \quad \tan \alpha = \frac{\sqrt{5}}{2}
\]

\[
\csc \alpha = 3 \quad \sec \alpha = \frac{3}{2} \quad \cot \alpha = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}
\]

Now we assign the appropriate signs to each of these values to find the values of the trigonometric functions of \(\theta\).

\[
\sin \theta = \frac{\sqrt{5}}{3} \quad \cos \theta = -\frac{2}{3} \quad \tan \theta = -\frac{\sqrt{5}}{2}
\]

\[
\csc \theta = 3 \quad \sec \theta = -\frac{3}{2} \quad \cot \theta = -\frac{2\sqrt{5}}{5}
\]
Finding the Exact Values of Trigonometric Functions

If \( \tan \theta = -4 \) and \( \sin \theta < 0 \), find the exact value of each of the remaining trigonometric functions of \( \theta \).

Solution

Since \( \tan \theta = -4 < 0 \) and \( \sin \theta < 0 \), it follows that \( \theta \) lies in quadrant IV. If \( \alpha \) is the reference angle for \( \theta \), then \( \tan \alpha = 4 = \frac{b}{a} \). With \( a = 1 \) and \( b = 4 \), we find \( r = \sqrt{1^2 + 4^2} = \sqrt{17} \). See Figure 64. Then

\[
\sin \alpha = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17} \quad \cos \alpha = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17} \quad \tan \alpha = \frac{4}{1} = 4
\]

\[
\csc \alpha = \frac{\sqrt{17}}{4} \quad \sec \alpha = \frac{\sqrt{17}}{1} = \sqrt{17} \quad \cot \alpha = \frac{1}{4}
\]

We assign the appropriate sign to each of these to obtain the values of the trigonometric functions of \( \theta \).

\[
\sin \theta = -\frac{4\sqrt{17}}{17} \quad \cos \theta = \frac{\sqrt{17}}{17} \quad \tan \theta = -4
\]

\[
\csc \theta = -\frac{\sqrt{17}}{4} \quad \sec \theta = \sqrt{17} \quad \cot \theta = -\frac{1}{4}
\]

Now Work Problem 99

### 7.4 Assess Your Understanding

#### Concepts and Vocabulary

1. For an angle \( \theta \) that lies in quadrant III, the trigonometric functions ___ and ___ are positive.
2. Two angles in standard position that have the same terminal side are ___.
3. The reference angle of 240° is ___.
4. True or False \( \sin 182° = \cos 2° \).
5. True or False \( \tan \frac{\pi}{2} \) is not defined.
6. True or False The reference angle is always an acute angle.
7. What is the reference angle of 600°?
8. In which quadrants is the cosine function positive?
9. If \( 0 < \theta < 2\pi \), for what angles \( \theta \), if any, is \( \tan \theta \) undefined?
10. What is the reference angle of \( \frac{13\pi}{3} \) ?

#### Skill Building

In Problems 11–20, a point on the terminal side of an angle \( \theta \) is given. Find the exact value of each of the six trigonometric functions of \( \theta \).

11. \((-3, 4)\)  
12. \((5, -12)\)  
13. \((2, -3)\)  
14. \((-1, -2)\)  
15. \((-3, -3)\)

16. \((2, -2)\)  
17. \((\sqrt{3}, 1)\)  
18. \((-1, \sqrt{3})\)  
19. \((\sqrt{2}, -\sqrt{2})\)  
20. \((-\sqrt{2}, -\sqrt{2})\)

In Problems 21–32, use a coterminal angle to find the exact value of each expression. Do not use a calculator.

21. \(\sin 405°\)  
22. \(\cos 420°\)  
23. \(\tan 405°\)  
24. \(\sin 390°\)  
25. \(\csc 450°\)  
26. \(\sec 540°\)

27. \(\cot 390°\)  
28. \(\sec 420°\)  
29. \(\cos \frac{33\pi}{4}\)  
30. \(\sin \frac{9\pi}{4}\)  
31. \(\tan(21\pi)\)  
32. \(\csc \frac{9\pi}{2}\)

In Problems 33–40, name the quadrant in which the angle \( \theta \) lies.

33. \(\sin \theta > 0, \cos \theta < 0\)  
34. \(\sin \theta < 0, \cos \theta > 0\)  
35. \(\sin \theta < 0, \tan \theta < 0\)

36. \(\cos \theta > 0, \tan \theta > 0\)  
37. \(\cos \theta > 0, \cot \theta < 0\)  
38. \(\sin \theta < 0, \cot \theta > 0\)

39. \(\sec \theta < 0, \tan \theta > 0\)  
40. \(\csc \theta > 0, \cot \theta < 0\)
In Problems 41–58, find the reference angle of each angle.

| 41. | −30° |
| 42. | 60° |
| 43. | 120° |
| 44. | 300° |
| 45. | 210° |
| 46. | 330° |
| 47. | \(\frac{5\pi}{4}\) |
| 48. | \(\frac{5\pi}{6}\) |
| 49. | \(\frac{8\pi}{3}\) |
| 50. | \(\frac{7\pi}{4}\) |
| 51. | −135° |
| 52. | −240° |
| 53. | −\(\frac{2\pi}{3}\) |
| 54. | −\(\frac{7\pi}{6}\) |
| 55. | 440° |
| 56. | 490° |
| 57. | \(\frac{15\pi}{4}\) |
| 58. | \(\frac{19\pi}{6}\) |

In Problems 59–88, use the reference angle to find the exact value of each expression. Do not use a calculator.

| 59. | \(\sin 150°\) |
| 60. | \(\cos 210°\) |
| 61. | \(\cos 315°\) |
| 62. | \(\sin 120°\) |
| 63. | \(\sin 510°\) |
| 64. | \(\cos 600°\) |
| 65. | \(\cos (-45°)\) |
| 66. | \(\sin (-240°)\) |
| 67. | \(\sec 240°\) |
| 68. | \(\csc 300°\) |
| 69. | \(\cot 330°\) |
| 70. | \(\tan 225°\) |
| 71. | \(\sin \left(\frac{3\pi}{4}\right)\) |
| 72. | \(\cos \left(\frac{2\pi}{3}\right)\) |
| 73. | \(\csc \left(\frac{\pi}{6}\right)\) |
| 74. | \(\sec \left(\frac{\pi}{4}\right)\) |
| 75. | \(\cos \frac{13\pi}{4}\) |
| 76. | \(\tan \frac{8\pi}{3}\) |
| 77. | \(\sin \left(-\frac{2\pi}{3}\right)\) |
| 78. | \(\cot \left(-\frac{\pi}{6}\right)\) |
| 79. | \(\tan \frac{14\pi}{3}\) |
| 80. | \(\sec \frac{11\pi}{4}\) |
| 81. | \(\csc (-315°)\) |
| 82. | \(\sec (-225°)\) |
| 83. | \(\sin (8\pi)\) |
| 84. | \(\cos (-2\pi)\) |
| 85. | \(\tan (7\pi)\) |
| 86. | \(\cot (5\pi)\) |
| 87. | \(\sec (-3\pi)\) |
| 88. | \(\csc \left(-\frac{5\pi}{2}\right)\) |

In Problems 89–106, find the exact value of each of the remaining trigonometric functions of \(\theta\).

| 89. | \(\sin \theta = \frac{12}{13}\), \(\theta\) in Quadrant II |
| 90. | \(\cos \theta = \frac{3}{5}\), \(\theta\) in Quadrant IV |
| 91. | \(\cos \theta = -\frac{4}{5}\), \(\theta\) in Quadrant III |
| 92. | \(\sin \theta = -\frac{5}{13}\), \(\theta\) in Quadrant III |
| 93. | \(\sin \theta = \frac{5}{13}\), \(90° < \theta < 180°\) |
| 94. | \(\cos \theta = \frac{4}{5}\), \(270° < \theta < 360°\) |
| 95. | \(\cos \theta = -\frac{1}{3}\), \(180° < \theta < 270°\) |
| 96. | \(\sin \theta = \frac{2}{3}\), \(180° < \theta < 270°\) |
| 97. | \(\sin \theta = \frac{2}{3}\), \(\tan \theta < 0\) |
| 98. | \(\cos \theta = -\frac{1}{4}\), \(\tan \theta > 0\) |
| 99. | \(\sec \theta = 2\), \(\sin \theta < 0\) |
| 100. | \(\csc \theta = 3\), \(\cot \theta < 0\) |
| 101. | \(\tan \theta = \frac{3}{4}\), \(\sin \theta < 0\) |
| 102. | \(\cot \theta = \frac{4}{3}\), \(\cos \theta < 0\) |
| 103. | \(\tan \theta = -\frac{1}{3}\), \(\sin \theta > 0\) |
| 104. | \(\sec \theta = -2\), \(\tan \theta > 0\) |
| 105. | \(\csc \theta = -2\), \(\tan \theta > 0\) |
| 106. | \(\cot \theta = -2\), \(\sec \theta > 0\) |

107. Find the exact value of \(\sin 40° + \sin 130° + \sin 220° + \sin 310°\).

108. Find the exact value of \(\tan 40° + \tan 140°\).

Applications and Extensions

109. If \(f(\theta) = \sin \theta = 0.2\), find \(f(\theta + \pi)\).

110. If \(g(\theta) = \cos \theta = 0.4\), find \(g(\theta + \pi)\).

111. If \(F(\theta) = \tan \theta = 3\), find \(F(\theta + \pi)\).

112. If \(G(\theta) = \cot \theta = -2\), find \(G(\theta + \pi)\).

113. If \(\sin \theta = \frac{1}{5}\), find \(\csc (\theta + \pi)\).

114. If \(\cos \theta = \frac{2}{3}\), find \(\sec (\theta + \pi)\).

115. Find the exact value of \(\sin 1° + \sin 2° + \sin 3° + \ldots + \sin 358° + \sin 359°\).

116. Find the exact value of \(\cos 1° + \cos 2° + \cos 3° + \ldots + \cos 358° + \cos 359°\).

117. Projectile Motion An object is propelled upward at an angle \(\theta, 45° < \theta < 90°\), to the horizontal with an initial velocity of \(v_0\) feet per second from the base of a plane that makes an angle of \(45°\) with the horizontal. See the illustration. If air resistance is ignored, the distance \(R\) that it travels up the inclined plane is given by the function

\[ R(\theta) = \frac{v_0^2 \sqrt{2}}{32} [\sin(2\theta) - \cos(2\theta) - 1] \]

(a) Find the distance \(R\) that the object travels along the inclined plane if the initial velocity is 32 feet per second and \(\theta = 60°\).

(b) Graph \(R = R(\theta)\) if the initial velocity is 32 feet per second.

(c) What value of \(\theta\) makes \(R\) largest?

Discussion and Writing

118. Give three examples that demonstrate how to use the theorem on reference angles.

119. Write a brief paragraph that explains how to quickly compute the value of the trigonometric functions of \(0°, 90°, 180°, \) and \(270°\).