Inverse Eigenvalue Problems
Constructing Matrices with Prescribed Eigenvalues

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Outline

Introduction
   Eigenvalues and Eigenvectors
   Inverse Eigenvalue Problems (IEP’s)

One Simple Algorithm
   Heuvers’ Algorithm
   Proof
   An Example
   Benefits and Drawbacks

Applications
What are Eigenvalues and Eigenvectors?

- An eigenvalue is “any number such that a given square matrix minus that number times the identity matrix has a zero determinant” [2].

\[ Ax = \lambda x \]
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- An eigenvalue is “any number such that a given square matrix minus that number times the identity matrix has a zero determinant” [2].
- $Ax = \lambda x$
Inverse Eigenvalue Problems (IEP’s)

- A well-studied yet continually developing branch of Linear Algebra concerning construction of matrices from spectral data.[3]
- Two basic components: solvability and computability.
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Konrad Heuvers’ Algorithm
Symmetric Matrices with Prescribed Eigenvalues and Eigenvectors

Let \( \{p_1, p_2, \ldots, p^n\} \) be an arbitrary orthonormal basis for \( \mathbb{R}^n \). These will become the eigenvectors.

Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be \( n \) arbitrary real numbers (the desired eigenvalues) and \( \tau \) be any real number such that \( \tau \leq \lambda_j \) for \( j = 1, 2, \ldots, n \).

Define \( \mu_j = \sqrt{\lambda_j - \tau} \) and \( b_j = \mu_j p_j \), and let \( B \) be the matrix comprised of the column vectors \( b_1, b_2, \ldots b_n \).

Let \( S \) be the matrix \( S = BB^T + \tau l \), a symmetric matrix with the above eigenvectors and eigenvalues.
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- Define \( \mu_j = \sqrt{\lambda_j - \tau} \) and \( b_j = \mu_j \mathbf{p}_j \), and let \( B \) be the matrix comprised of the column vectors \( \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \).
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Proof of Heuvers’ Algorithm

- The columns of \( B \) are \((b_1, b_2, \ldots, b_n) = (\mu_1 p_1, \mu_2 p_2, \ldots, \mu_n p_n)\).
- The rows of \( B^T \) are of the form \( \mu_i p_i^T \).
- It must be shown that \( Sp_j = \lambda_j p_j \).
Proof of Heuvers’ Algorithm

- The columns of $B$ are 
  $$(b_1, b_2, \ldots, b_n) = (\mu_1 p_1, \mu_2 p_2, \ldots, \mu_n p_n).$$
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Proof of Heuvers’ Algorithm

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- The rows of $B^T$ are of the form $\mu_i p_i^T$.
- It must be shown that $Sp_j = \lambda_j p_j$. 
Proof (cont., 2)

\[ S p_j = (BB^T + \tau I)p_j \]
\[ = BB^T p_j + \tau p_j \]
\[ = [\mu_1 p_1, \mu_2 p_2, \ldots, \mu_n p_n] \begin{bmatrix} \mu_1 p_1^T p_j \\ \mu_2 p_2^T p_j \\ \vdots \\ \mu_n p_n^T p_j \end{bmatrix} + \tau p_j \]
Proof

Proof (cont., 3)

- Column vector all zeros except $p_j$ dotted with itself is one.

\[ \begin{align*}
\cdots &= [\mu_1p_1, \mu_2p_2, \ldots, \mu_np_n] \\
&\begin{bmatrix}
\mu_1p_1^T p_j \\
\mu_2p_2^T p_j \\
\vdots \\
\mu_np_n^T p_j
\end{bmatrix} + \tau p_j \\
&= [\mu_1p_1, \mu_2p_2, \ldots, \mu_np_n] \\
&\begin{bmatrix}
0 \\
\mu_j \\
\vdots \\
0
\end{bmatrix} + \tau p_j
\end{align*} \]
Proof (cont., 3)

- Column vector all zeros except \( p_j \) dotted with itself is one.

\[
\ldots = [\mu_1 p_1, \mu_2 p_2, \ldots, \mu_n p_n] \begin{bmatrix}
\mu_1 p_j^T p_j \\
\mu_2 p_2^T p_j \\
\vdots \\
\mu_n p_n^T p_j
\end{bmatrix} + \tau p_j
\]

\[
= [\mu_1 p_1, \mu_2 p_2, \ldots, \mu_n p_n] \begin{bmatrix}
0 \\
\mu_j \\
\vdots \\
0
\end{bmatrix} + \tau p_j
\]
Proof (cont., 4)

We have shown that $Sp_j = \lambda_j p_j$, therefore each vector $p_j$ and corresponding scalar $\lambda_j$ are an eigenvector and eigenvalue for the matrix $S$. 

\[
\begin{align*}
= & \mu_j^2 p_j + \tau p_j \\
= & (\mu_j^2 + \tau)p_j \\
= & \lambda_j p_j
\end{align*}
\]
Proof (cont., 4)

We have shown that $Sp_j = \lambda_j p_j$, therefore each vector $p_j$ and corresponding scalar $\lambda_j$ are an eigenvector and eigenvalue for the matrix $S$. 

\[
\begin{align*}
\mu_j^2 p_j + \tau p_j &= (\mu_j^2 + \tau)p_j \\
&= \lambda_j p_j
\end{align*}
\]
An Example

- Arbitrary orthonormal basis for $\mathbb{R}^2$:

$$B_{\mathbb{R}^2} = \left\{ \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \right\}$$

- For simplicity of computing $\mu_1$ and $\mu_2$, we’ll choose $\lambda_1 = 2$, $\lambda_2 = 5$, and $\tau = 1$.

$$\mu_1 = \sqrt{\lambda_1 - \tau} = \sqrt{2 - 1} = 1$$

$$\mu_2 = \sqrt{\lambda_2 - \tau} = \sqrt{5 - 1} = 2$$
An Example

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\begin{align*}
\mu_1 &= \sqrt{\lambda_1 - \tau} = \sqrt{2 - 1} = 1 \\
\mu_2 &= \sqrt{\lambda_2 - \tau} = \sqrt{5 - 1} = 2
\end{align*}
\]
An Example

- Arbitrary orthonormal basis for $\mathbb{R}^2$:

$$\mathcal{B}_{\mathbb{R}^2} = \left\{ \begin{bmatrix} \sqrt{2}/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{2}/2 \end{bmatrix} \right\}$$

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$$\begin{align*}
\mu_1 &= \sqrt{\lambda_1 - \tau} = \sqrt{2 - 1} = 1 \\
\mu_2 &= \sqrt{\lambda_2 - \tau} = \sqrt{5 - 1} = 2
\end{align*}$$
Create the matrix $B$, composed of the columns $b_1$ and $b_2$:

$$B = [b_1, b_2]$$

$$= [\mu_1 p_1, \mu_2 p_2]$$

$$= \begin{bmatrix} 1 \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, 2 \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2} \\ \sqrt{2}/2 & \sqrt{2} \end{bmatrix}$$
An Example (cont., 2)

Create the matrix $B$, composed of the columns $b_1$ and $b_2$:

$$B = [b_1, b_2]$$

$$= [\mu_1 p_1, \mu_2 p_2]$$

$$= \begin{bmatrix} 1 \begin{bmatrix} \sqrt{2}/2 \end{bmatrix} & 2 \begin{bmatrix} -\sqrt{2}/2 \end{bmatrix} \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2} \\ \sqrt{2}/2 & \sqrt{2} \end{bmatrix}$$
Now we can create our matrix $S$:

$$S = BB^T + \tau I$$

$$= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2} \\ \sqrt{2}/2 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2} & \sqrt{2} \end{bmatrix} + 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5/2 & -3/2 \\ -3/2 & 5/2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7/2 & -3/2 \\ 3/2 & 7/2 \end{bmatrix}$$
An Example (cont., 4)

Check this solution by first finding the eigenvalues of $S$:

$$
\det(S - \lambda I) = 0
$$

$$
\begin{vmatrix}
\frac{7}{2} - \lambda & -\frac{3}{2} \\
-\frac{3}{2} & \frac{7}{2} - \lambda
\end{vmatrix} = 0
$$

$$(7/2 - \lambda)^2 - (-3/2)^2 = 0$$

$$49/4 - 7\lambda + \lambda^2 - 9/4 = 0$$

$$\lambda^2 - 7\lambda + 10 = 0$$

$$(\lambda - 2)(\lambda - 5) = 0$$

$$\lambda = 2, 5$$
...and then by finding the corresponding eigenvectors:

\[(S - 2I)x = 0\]

\[
\begin{bmatrix}
3/2 & -3/2 \\
-3/2 & 3/2 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
x \\
y \\
\end{bmatrix} = \begin{bmatrix}
1 \\
1 \\
\end{bmatrix}
\]

\[(S - 5I)x = 0\]

\[
\begin{bmatrix}
-3/2 & -3/2 \\
-3/2 & -3/2 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
x \\
y \\
\end{bmatrix} = \begin{bmatrix}
-1 \\
1 \\
\end{bmatrix}
\]
An Example (cont., 5)

...and then by finding the corresponding eigenvectors:

\[
(S - 2I)x = 0
\]
\[
\begin{bmatrix}
3/2 & -3/2 \\
-3/2 & 3/2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\]

\[
(S - 5I)x = 0
\]
\[
\begin{bmatrix}
-3/2 & -3/2 \\
-3/2 & -3/2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
\[
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
1 \\
-1
\end{bmatrix}
\]
Note that the eigenvectors of the $S$ we created with the algorithm are scalar multiples of the eigenvectors we chose beforehand.

Since the nullspaces of $S - 2I$ and $S - 5I$ are both closed under scalar multiplication, the eigenvectors we found confirm the validity of the algorithm.
An Example (cont., 6)

- Note that the eigenvectors of the $S$ we created with the algorithm are scalar multiples of the eigenvectors we chose beforehand.

- Since the nullspaces of $S - 2I$ and $S - 5I$ are both closed under scalar multiplication, the eigenvectors we found confirm the validity of the algorithm.
An Example (cont., 7)

Figure: Eigenvectors of matrix $S$. 
Benefits and Drawbacks of Heuvers’ Algorithm

- Simple to understand and compute.
- Always creates symmetric matrices, must normalize eigenvectors first.
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- Always creates symmetric matrices, must normalize eigenvectors first.
Applications of Inverse Eigenvalue Problems

- Found in applications where goal is finding physical parameters of a system based on known behavior or constructing a system with physical parameters resulting in a desired dynamical behavior [3].
  - Particle physics
  - Molecular spectroscopy
  - Geophysics
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For Further Reading

G. Strang. 
*Introduction to Linear Algebra, Fourth Edition.*

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*WordNet A Lexical Database for English, 2010*
http://wordnetweb.princeton.edu/perl/webwn?
s=eigenvalue
For Further Reading II

M. Chu, G. Golub. 
Inverse Eigenvalue Problems: Theory and Applications. 
*Department of Mathematics, North Carolina State University, 2001*
http://www4.ncsu.edu/~mtchu/Research/Lectures/Iep/preface.ps

K. Heuvers. 
Symmetric Matrices with Prescribed Eigenvalues and Eigenvectors 