The Maclaurin Trisectrix

Justin Seago

December 8, 2008

History

In 1742 Colin Maclaurin investigated the curve now known as the *Maclaurin trisectrix* in an effort to solve one of the geometric problems of antiquity: that of trisecting an arbitrary angle. The Greeks sought to trisect an angle using only a compass and unmarked straight edge; however, in 1835 Wantzel proved this was impossible.[7] That is not to say it is altogether impossible to trisect any angle—on the contrary, there are many ways to go about it.[2] There are, however, many individuals who claim to have trisected an angle when they find an approximation and do not see any reason for proving their assertion with mathematical rigor. For many practical applications that's just fine, but any mathematician worth his or her salt knows better than to assume that a result is correct just because it seems to be correct. Maclaurin was certainly a worthy mathematician, and as we will see, he discovered a nice little method for precisely trisecting angles.

Before we jump into the mathematics of the Maclaurin trisectrix, there are a few things to be said of Mr. Maclaurin. He was born in Kilmodan, Scotland in 1698. By the time he was ten years old, both of his parents had died and his uncle Daniel Maclaurin was charged with his upbringing. Maclaurin excelled as a student and entered the University of Glasgow at the age of 11 and received his M.A. when he was 14. At 19, he was elected for professorship at Marischal College in the University of Aberdeen.[5] For this Maclaurin reportedly held a world record as the youngest professor until April 2008.[4]

He met Isaac Newton in 1719 on a trip to London and later wrote a two-volume work entitled *A Treatise on Fluxions* (1742) wherein he defended Newton's calculus.[3][5] Other significant publications by Maclaurin include *Geometria Organica* (1720), *A Treatise of Algebra* (1748), and *An Account of Sir Isaac Newton's Philosophy* (1748).[6]

Maclaurin spent most of his life a bachelor until he married Anne Stewart in 1725, with whom he had seven children. Maclaurin died of dropsy on the June 14th, 1746. Two of his aforementioned works were published posthumously. He is best remembered today for his work on the special Taylor Series centered about zero that bears his name.[5]

Equations of the Maclaurin Trisectrix

With the asymptote at $x = a$, origin at crunode

Implicit Cartesian

$$y^2 = \frac{x^2\,(x + 3\,a)}{a - x}$$

Parametric

$$x = a\,\frac{t^2 - 3}{t^2 + 1} \quad , \quad y = a\,\frac{t\,(t^2 - 3)}{t^2 + 1}$$

Polar

$$r = -\frac{2\,a\,\sin(3\,\theta)}{\sin(2\,\theta)}$$

With the asymptote at $x = 3\,a$

Polar

$$r = -a\,\sec\left(\frac{1}{3}\,\theta\right)$$ [8]

With asymptote at $x = \frac{1}{2}\,a$

Implicit Cartesian

$$x^3 + xy^3 = \frac{1}{2}\,a\,(y^2 - 3\,x^2)$$

Polar

$$r = -\frac{1}{2}\,a(4\cos(\theta) - \sec(\theta)) = -a\,\frac{\sin(3\theta)}{\sin(2\theta)}$$

Parametric

$$x = -\frac{1}{2}\,a\,\left(4\cos^2(\theta) - 1\right) \quad , \quad y = -\frac{1}{2}\,a\,\left(4\cos^2(\theta) - 1\right)\tan(\theta)$$ [1]

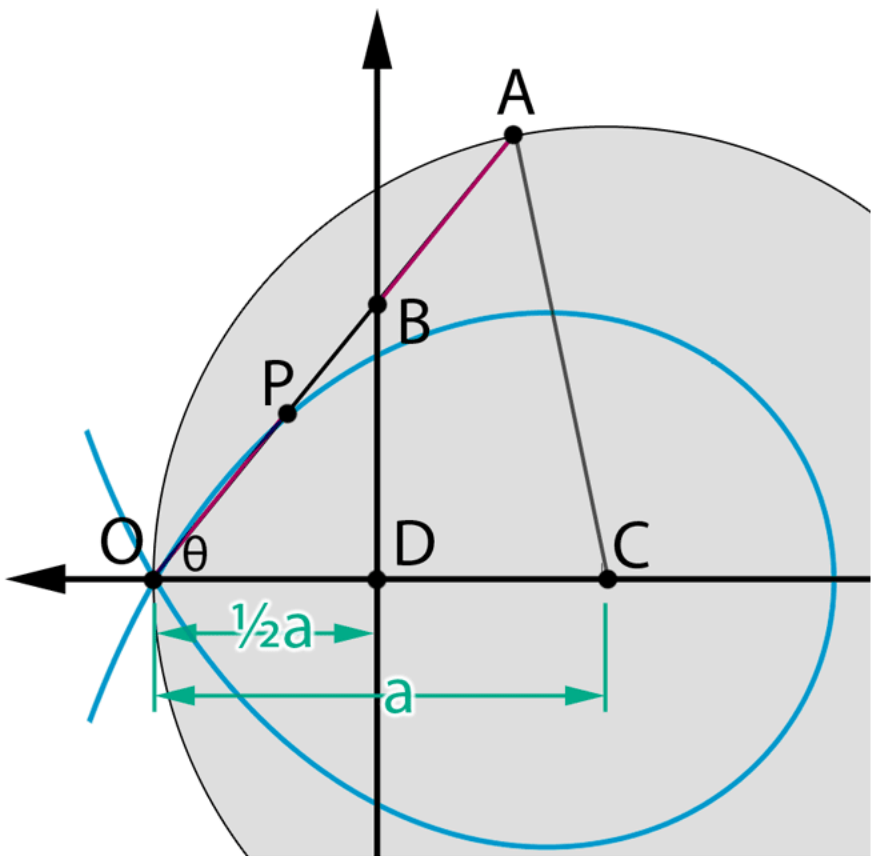
Definition

1. Begin with a circle centered at a point ***C*** with a radius ***CO*** along the x-axis.

2. Bisect the radius ***CO*** with a point ***D*** and run a vertical line in the y-direction through ***D***.

2. Draw a line ***OBA*** from ***O*** intersecting the vertical line at ***B*** and the circle at ***A***.

3. On the line segment ***OA***, create a point ***P*** that is the same distance from ***O*** as the distance ***AB***. That is, place ***P*** on ***OA*** such that ***OP* = *AB***. The locus of the point ***P*** as ***A*** traces the circumference of the circle is the Maclaurin trisectrix.[1]*



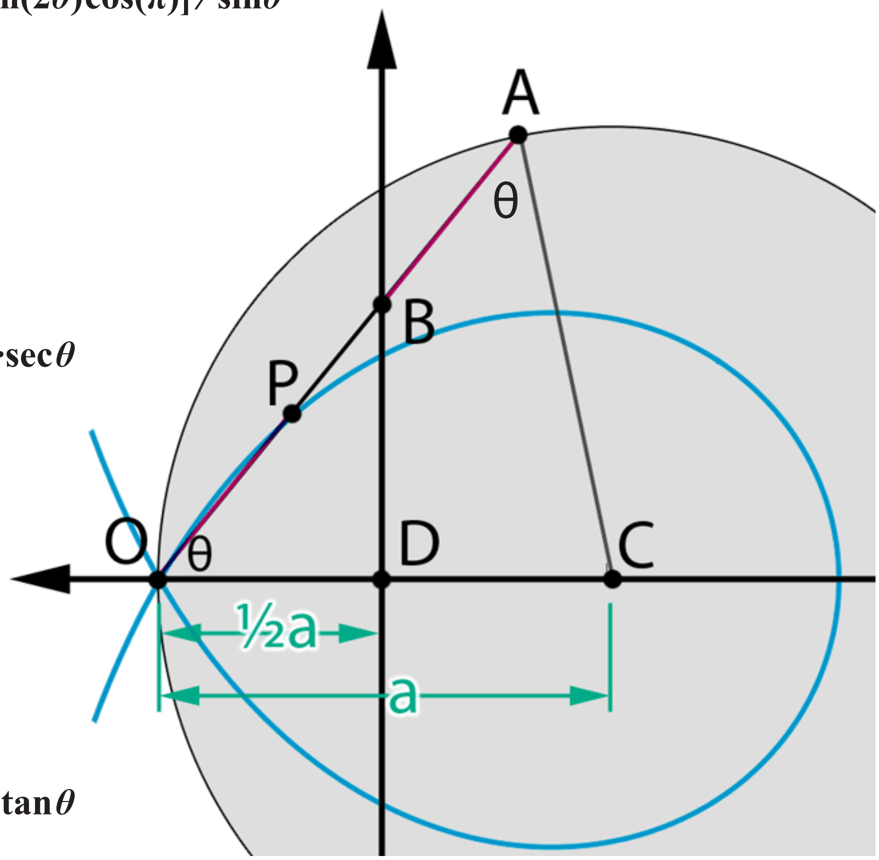
* Note that I have chosen to make the trisectrix flipped about the x and y axes from the forms mentioned on the preceeding page. This oddly seems unconventional, despite the fact that it allows one avoid several negative signs in the resulting equations.

Parametrizing

Let $\boldsymbol{OC} = \boldsymbol{a}$ and $\angle \boldsymbol{AOC} = \boldsymbol{\theta}$, then $\boldsymbol{x} = \boldsymbol{OP}\cos\boldsymbol{\theta} = \boldsymbol{AB}\cos\boldsymbol{\theta}$
Also $\boldsymbol{AB} = \boldsymbol{OA} - \boldsymbol{OB}$

If we let $\boldsymbol{OA} = \boldsymbol{c}$, then by the law of sines

$$\tfrac{1}{a}\cdot\sin\theta = \tfrac{1}{c}\cdot\sin(\pi - 2\theta)$$
$$\Rightarrow\quad c = a[\sin(\pi - 2\theta)]/\sin\theta$$
$$= a[\sin(\pi)\cos(2\theta) - \sin(2\theta)\cos(\pi)]/\sin\theta$$
$$= a\cdot\sin(2\theta)/\sin\theta$$
$$= 2a\cdot\cos\theta$$



Let $\boldsymbol{OB} = \boldsymbol{b}$, then

$$b\cdot\cos\theta = \tfrac{1}{2}a$$
$$\Rightarrow\quad b = \tfrac{1}{2}a\cdot\sec\theta$$

Hence, $\boldsymbol{AB} = \boldsymbol{c} - \boldsymbol{b} = 2\boldsymbol{a}\cdot\cos\boldsymbol{\theta} - \tfrac{1}{2}\boldsymbol{a}\cdot\sec\boldsymbol{\theta}$

$$x = AB\cos\theta$$
$$= (2a\cdot\cos\theta - \tfrac{1}{2}a\cdot\sec\theta)\cos\theta$$
$$= 2a\cdot\cos^2\theta - \tfrac{1}{2}a$$
$$= \tfrac{1}{2}a(4\cos^2\theta - 1)$$

$$y = AB\sin\theta$$
$$= (2a\cdot\cos\theta - \tfrac{1}{2}a\cdot\sec\theta)\sin\theta$$
$$= 2a\cdot\cos(\theta)\sin(\theta) - \tfrac{1}{2}a\cdot\tan\theta$$
$$= \tfrac{1}{2}a[4\cos^2(\theta)\sin(\theta)/\sin(\theta) - 1]\tan\theta$$
$$= \tfrac{1}{2}a(4\cos^2\theta - 1)\tan\theta$$

Thus,

$$x = \tfrac{1}{2}a(4\cos^2\theta - 1)$$
$$y = \tfrac{1}{2}a(4\cos^2\theta - 1)\tan\theta$$

are the parametric equations of the Maclaurin trisectrix with respect to $\boldsymbol{\theta}$.

Polar and Cartesian Derivations

From the parametric equations we may easily deduce a polar equation since
$\boldsymbol{x = \mathbf{r}\cdot\cos\theta}$
$\Rightarrow\ \ \boldsymbol{\mathbf{r}\cdot\cos\theta = \tfrac{1}{2}a(4\cos^2\theta - 1)}$
$\Rightarrow\ \ \boldsymbol{\mathbf{r} = \tfrac{1}{2}a(4\cos\theta - \sec\theta)}$ which is the polar equation of the trisectrix.

To find the Cartesian equation, note that $\mathbf{r} = \sqrt{x^2 + y^2}$ by the Pythagorean theorem. So
$\boldsymbol{x = \mathbf{r}\cdot\cos\theta \Leftrightarrow x/\sqrt{x^2+y^2} = \cos\theta}$.
$\Rightarrow\ \ \boldsymbol{\sqrt{x^2+y^2} = \tfrac{1}{2}a[4x/\sqrt{x^2+y^2} - \sqrt{x^2+y^2}/x]}$

Multiplying both sides by $\sqrt{x^2+y^2}$ we get
$\boldsymbol{x^2 + y^2 = \tfrac{1}{2}a[4x - (x^2+y^2)/x]}$
$\Rightarrow\ \ \boldsymbol{x^3 + xy^2 = \tfrac{1}{2}a(4x^2 - x^2 - y^2)}$
$\Rightarrow\ \ \boldsymbol{x^3 + xy^2 = \tfrac{1}{2}a(3x^2 - y^2)}$ which is the implicit Cartesian equation of the Maclaurin trisectrix.

Trisection Proof

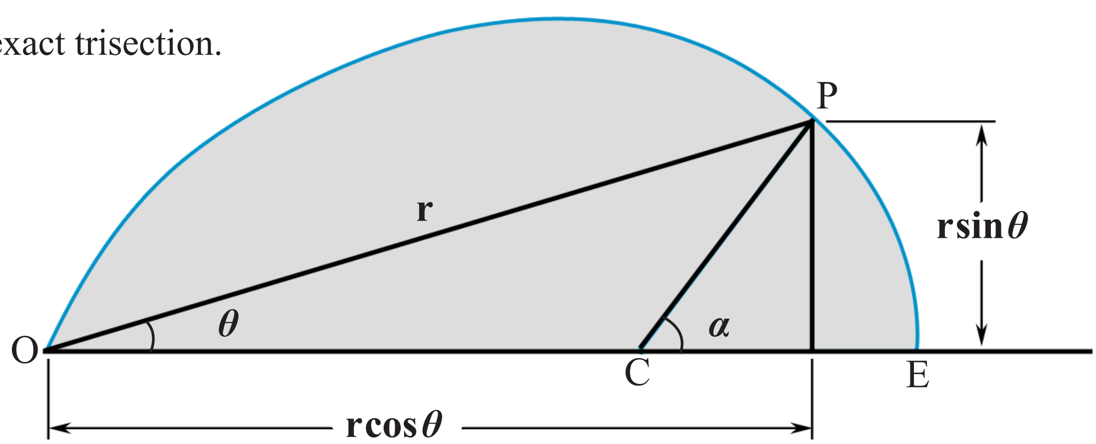
Let $\boldsymbol{OC = 1}$ (i.e., $\boldsymbol{a = 1}$), $\boldsymbol{\angle EOP = \theta}$, and $\boldsymbol{\angle ECP = \alpha}$. We must show that $\boldsymbol{\theta = \tfrac{1}{3}\alpha}$.

Note that $\boldsymbol{\tan\alpha = \mathbf{r}\sin(\theta)/(\mathbf{r}\cos\theta - 1)}$.

Recall from the polar equation of the trisectrix that $\boldsymbol{\mathbf{r} = \tfrac{1}{2}a(4\cos\theta - \sec\theta)}$ which implies that $\boldsymbol{\tan\alpha}$ is equal to $\boldsymbol{\tan 3\theta}$ after some difficult algebraic manipulation—I used Maple to verify this:

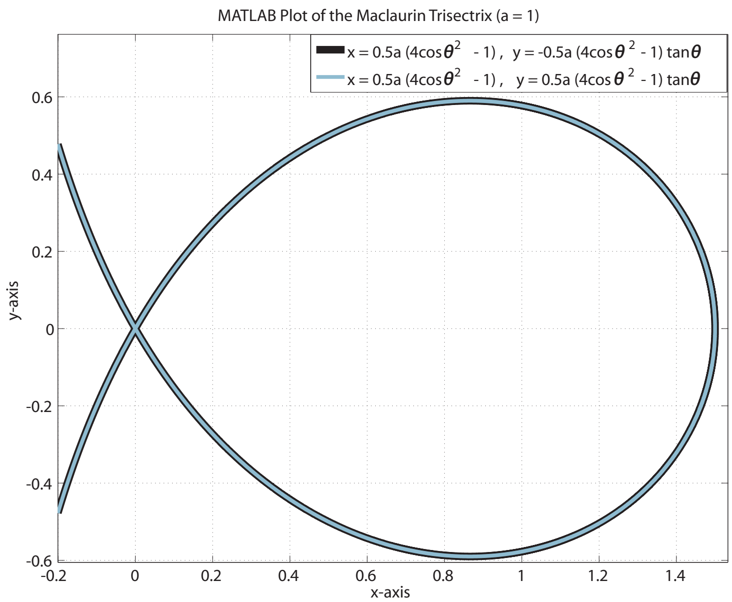
$$\frac{\frac{1}{2}(4\cos(\theta) - \sec(\theta))\cdot\sin(\theta)}{\frac{1}{2}(4\cos(\theta) - \sec(\theta))\cdot\cos(\theta) - 1} = \tan(3\theta) \xrightarrow{\text{test relation}} \textit{true}$$

Thus, $\boldsymbol{\theta = \tfrac{1}{3}\alpha}$ an exact trisection.



Other Properties

The Maclaurin trisectrix is an anallagmatic curve, meaning it is the same when it is inverted. Here I have plotted the trisectrix for values of the form (x, y) and (x, -y) with the plot (x, -y) in black having a *LineWidth* of 8 points and the (x, y) plot in blue with a *LineWidth* of 4 points. This is a vector graphic so you should be able too zoom in a bit for a closer look.



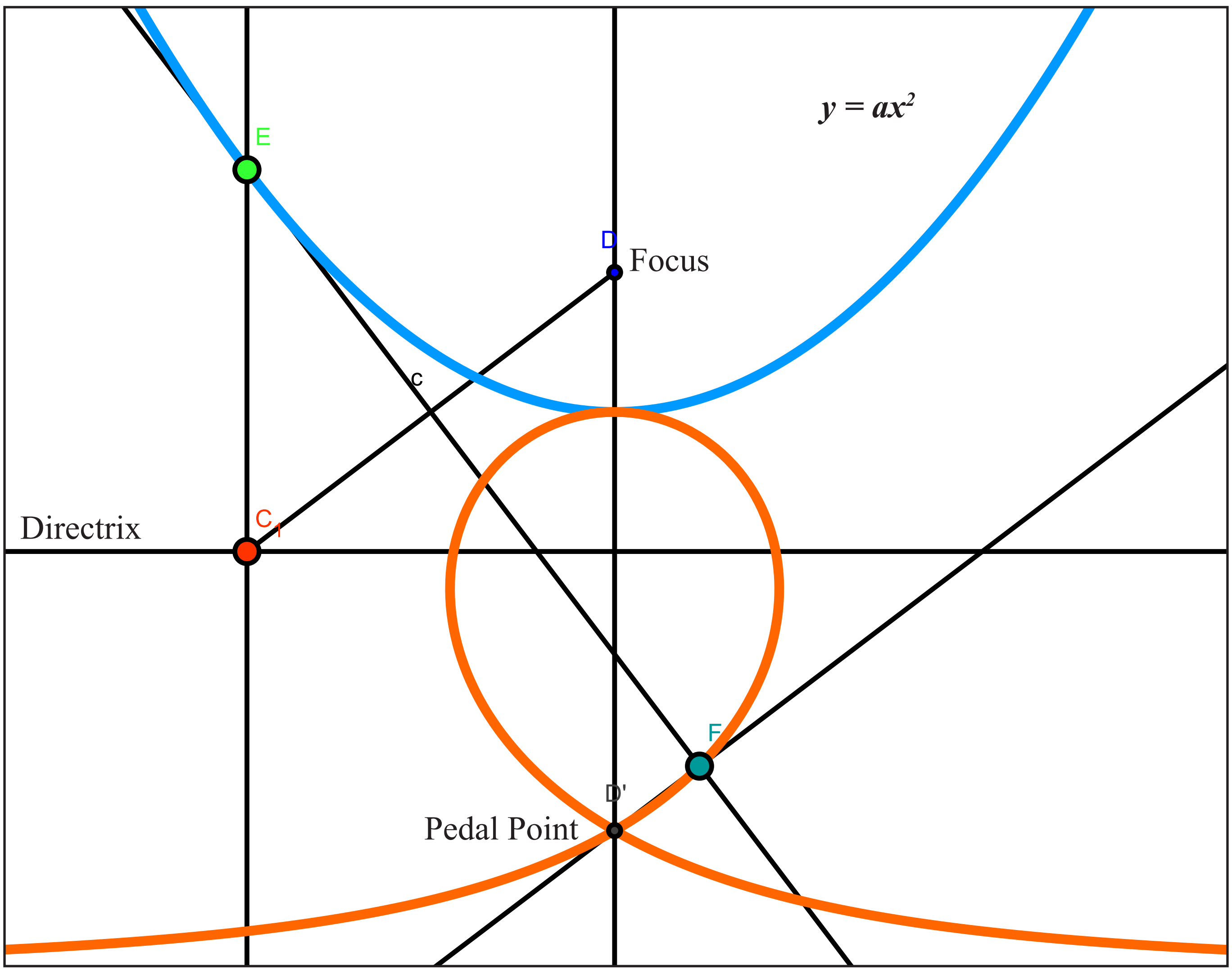
The MATLAB code that produced the image above:

```
clc
close all
%% (x,-y)
theta=linspace(-pi/3-0.15,pi/3+0.15,10000);
a=1;
x=0.5*a*(4*cos(theta).^2-1);
y=-0.5*a*(4*cos(theta).^2-1).*tan(theta);
plot(x,y,'LineWidth',8,'Color','k')
% comet(x,y)
hold on

%% (x, y)
x=0.5*a*(4*cos(theta).^2-1);
y=0.5*a*(4*cos(theta).^2-1).*tan(theta);
plot(x,y,'LineWidth',4,'Color',[.5,.8,.8])
% comet(x,y)

legend('x = 0.5a (4cos^2\theta - 1) ,   y = -0.5a (4cos^2\theta - 1) tan\theta',...
   'x = 0.5a (4cos^2\theta - 1) ,    y = 0.5a (4cos^2\theta - 1) tan\theta')
title('MATLAB Plot of the Maclaurin Trisectrix (a = 1)')
xlabel('x-axis')
ylabel('y-axis')
grid on
axis equal
```

As a Pedal Curve

The Maclaurin Trisectrix is the pedal curve of a parabola with a pedal point mirroring the focus of the parabola across the directrix.

Note the asymptote, which is the same distance (½***a***) from the double point of the trisectrix as the focus of the parabola is from its vertex.



Area of the Loop

$$x = \frac{1}{2}a\left(4\cos^2(\theta) - 1\right)$$

$$y = \frac{1}{2}a\left(4\cos^2(\theta) - 1\right)\tan(\theta)$$

$$x'(\theta) = 4a\cdot\cos(\theta)\sin(\theta)$$

$$A = \int_{\alpha}^{\beta} y(\theta)x'(\theta)\,d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\frac{1}{2}a\left(4\cos^2(\theta) - 1\right)\tan(\theta)\cdot 4a\cdot\cos(\theta)\sin(\theta)\right)d\theta$$

$$= 2a^2\cdot\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(4\cos^3(\theta)\sin(\theta) - \cos(\theta)\sin(\theta)\right)\tan(\theta)\,d\theta$$

$$= 4a^2\cdot\int_{0}^{\frac{\pi}{3}} \left(4\cos^2(\theta)\sin^2(\theta) - \sin^2(\theta)\right)d\theta$$

$$= 4a^2\cdot\int_{0}^{\frac{\pi}{3}} \left(4\left(\frac{1}{2}(1 + \cos(2\theta))\right)\left(\frac{1}{2}(1 - \cos(2\theta))\right) - \frac{1}{2}(1 - \cos(2\theta))\right)d\theta$$

$$= 4a^2\cdot\int_{0}^{\frac{\pi}{3}} \left(1 - \cos^2(2\theta) - \frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)d\theta$$

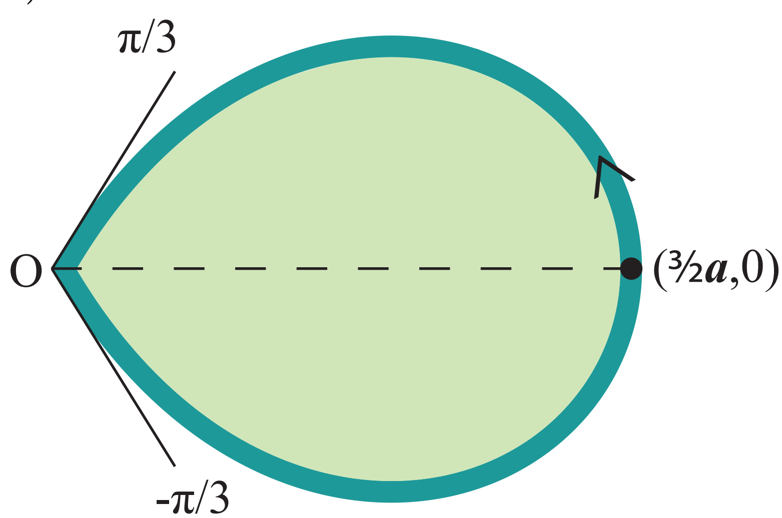
$$= 4a^2\cdot\int_{0}^{\frac{\pi}{3}} \left(\frac{1}{2} - \frac{1}{2}(1 + \cos(4\theta)) + \frac{1}{2}\cos(2\theta)\right)d\theta$$

$$= 2a^2\cdot\int_{0}^{\frac{\pi}{3}} \left(-\cos(4\theta) + \cos(2\theta)\right)d\theta$$

$$= 2a^2\cdot\left[-\frac{1}{4}\sin(4\theta) + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{3}}$$

$$= a^2\cdot\left[-\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}\right]$$

$$= a^2\cdot\frac{3\sqrt{3}}{4}$$



References

[1] Basset, A. B. (1901), *An Elementary Treatise on Cubic and Quartic Curves*, 81, http://books.google.com/books?id=yUxtAAAAMAAJ

[2] Loy, Jim. *Trisection of an Angle*. http://www.jimloy.com/geometry/trisect.htm

[3] MacLaurin, Colin (1801), *A Treatise on Fluxions: In Two Volumes*, http://books.google.com/books?id=QOg3AAAAMAAJ

[4] McNeill, David (2008), *University appoints world's youngest professor*, http://www.independent.co.uk/news/world/asia/university-appoints-child-prodigy-worlds-youngest-professor-818776.html

[5] O'Connor, J. J. and Robertson, E. F. (1999). *Maclaurin biography*. http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Maclaurin.html

[6] O'Connor, J. J. and Robertson, E. F. (2007). *Mathematical Works of Colin Maclaurin.* http://www-groups.dcs.st-and.ac.uk/~history/Extras/Maclaurin_publications.html

[7] Weisstein, Eric W. *Angle Trisection.* From MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/AngleTrisection.html

[8] Weisstein, Eric W. *Maclaurin Trisectrix.* From MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/MaclaurinTrisectrix.html