Tacoma Narrows Bridge

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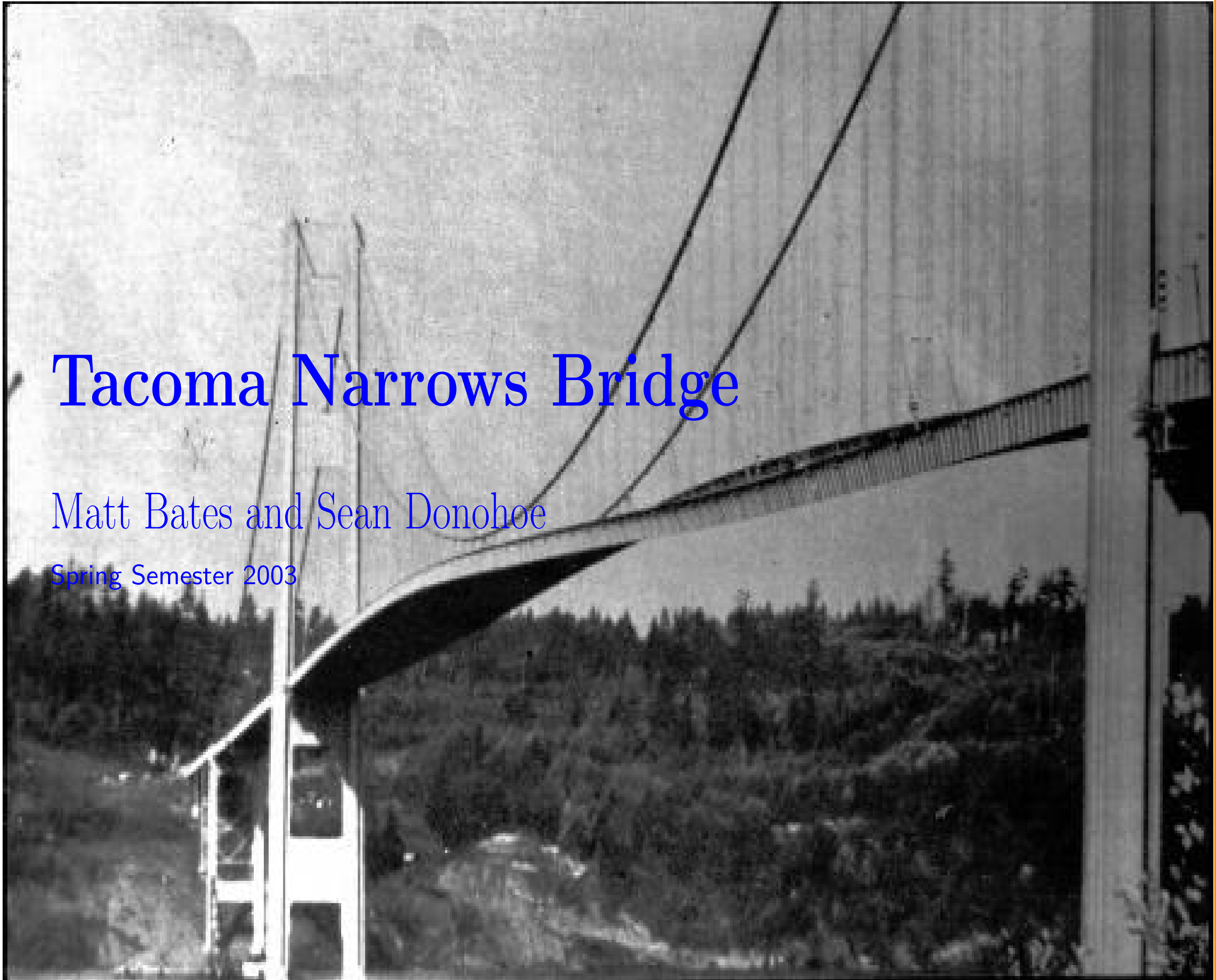


Introduction

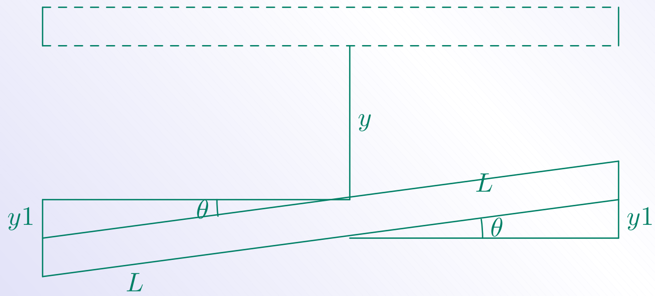
Before the collapse of the Tacoma Narrows Bridge suspension bridges were commonly constructed using rule of thumb. This method proved dangerous, with many of these bridges exhibiting odd behavior consisting of large vertical and torsional oscillations. In the case of Tacoma Narrows, it is believed that not the vertical oscillations but the torsional oscillations ultimately led to its destruction. We can model these torsional oscillations with two different equations. These 2 equations will derived from the kinetic and potential energy equations for the bridge.

PE and KE energy of the rod

Adding the physics equations for vertical kinetic energy and rotational energy we get the total kinetic energy.

$$T = \frac{1}{2}mv^2 + \frac{1}{6}mL^2\dot{\theta}^2 \tag{1}$$

Modelling a cross section of the bridge (with length $2L$) will help us visualize how we get our potential energy.



There is a vertical deflection on both sides due to torsional oscillations.

On the left side the vertical deflection is:

$$y_t = (y + y_1)^+, \tag{2}$$

and on the right the vertical deflection is:

$$y_t = (y - y_1)^+. \tag{3}$$

The $+$ exponet refers to the fact that the cables only act like springs when pulled in the downward direction.

Looking at the picture depicting total displacement of a cross section of the bridge we see that y_1 is:

$$L \sin \theta, \tag{4}$$

Subbing this into the y_t equations above we get:

$$\begin{aligned} y_t &= (y - L \sin \theta)^+ \\ y_t &= (y + L \sin \theta)^+. \end{aligned} \tag{5}$$

From physics the total potential energy is:

$$PE = \frac{1}{2}ky^2 - mgy. \tag{6}$$

Plugging y_t in for y we get:

$$PE = V = \frac{k}{2}[((y - L\sin\theta)^2)^+ + ((y + L\sin\theta)^2)^+] - mgy \tag{7}$$

Energy to oscillation equations

Using the Lagrangian $\mathcal{L} = T - V$ we get:

$$\mathcal{L} = \frac{m\dot{y}^2}{2} + \frac{1}{6}mL^2\dot{\theta}^2 - \frac{1}{2}k[((y - L\sin\theta)^2)^+ + ((y + L\sin\theta)^2)^+] + mgy \tag{8}$$

Applying the Euler Lagrange equation

$$\frac{d}{dt}\left(\frac{\delta\mathcal{L}}{\delta\dot{\theta}}\right) = \frac{\delta\mathcal{L}}{\delta\theta} \tag{9}$$

and taking the partial derivatives we get:

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta\theta} &= \frac{1}{2}k[2(y - L\sin\theta)^+(-L\cos\theta) + 2(y + L\sin\theta)^+(L\cos\theta)] \\ &= kL\cos\theta[(y - L\sin\theta)^+ - (y + L\sin\theta)^+]\end{aligned} \tag{10}$$

and

$$\frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \dot{\theta}} = \frac{1}{3} m L^2 \ddot{\theta}. \tag{11}$$

With some intense simplifications we come to:

$$\ddot{\theta} = \frac{3k}{mL} \cos\theta [(y - L\sin\theta)^+ - (y + L\sin\theta)^+] \tag{12}$$

Applying the Euler Lagrange equation

$$\frac{d}{dt}\left(\frac{\delta \mathcal{L}}{\delta \dot{y}}\right) = \frac{\delta \mathcal{L}}{\delta y}, \tag{13}$$

and taking the partial derivatives we get:

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta y} &= -\frac{1}{2}k[2(y - L\sin\theta)^{+} + 2(y + L\sin\theta)^{+}] + mg \\ &= -k[(y - L\sin\theta)^{+} + (y + L\sin\theta)^{+}] + mg \end{aligned} \tag{14}$$

and

$$\frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \dot{y}} = \frac{d}{dt}(m\dot{y}) = m\ddot{y}. \tag{15}$$

With some intense simplifications we get:

$$\ddot{y} = -\frac{k}{m}[(y - Lsin\theta)^{+} + (y + L\sin\theta)^{+}] + g. \tag{16}$$

We then add a damping term, $\delta\dot{\theta}$ and $\delta\dot{y}$, and we also add a external forcing term f(t) to the torsional equation. Doing this we get:

$$\ddot{\theta} = -\delta\dot{\theta} + \frac{3k}{mL}\cos\theta[(y - L\sin\theta)^{+} - (y + L\sin\theta)^{+}] + f(t) \qquad (17)$$

$$\ddot{y} = -\delta\dot{y} - \frac{k}{m}[(y - Lsin\theta)^{+} + (y + L\sin\theta)^{+}] + g. \qquad (18)$$

Simplifying the torsional equation $\ddot{\theta}$:

$$\begin{aligned}
\ddot{\theta} &= -\delta\dot{\theta} + \frac{3k}{mL}\cos\theta[(y - L\sin\theta) - (y + L\sin\theta)] + f(t) \\
\ddot{\theta} &= -\delta\dot{\theta} + \frac{3k}{mL}\cos\theta[y - L\sin\theta - y - L\sin\theta] + f(t) \\
\ddot{\theta} &= -\delta\dot{\theta} + \frac{3k}{mL}\cos\theta(-2L\sin\theta) + f(t) \\
\ddot{\theta} &= -\delta\dot{\theta} - \frac{6k}{m}\cos\theta(\sin\theta) + f(t). \qquad (19)
\end{aligned}$$

Torsional oscillations

The one we are really interested in is the $\ddot{\theta}$ equation because this is what lead to the collapse of the bridge. Assuming that the cables never lose tension and simplifying this equation we get:

$$\ddot{\theta} = -\delta\dot{\theta} - \frac{6k}{m}\cos\theta\sin\theta + f(t) \tag{20}$$

This is the non-linear equation. We will then assume small oscillations and say $sin\theta$ is approximately θ and $cos\theta$ is approximately 1 to linearize it to get:

$$\ddot{\theta} = -\delta\dot{\theta} - \frac{6k}{m}\theta + f(t) \tag{21}$$

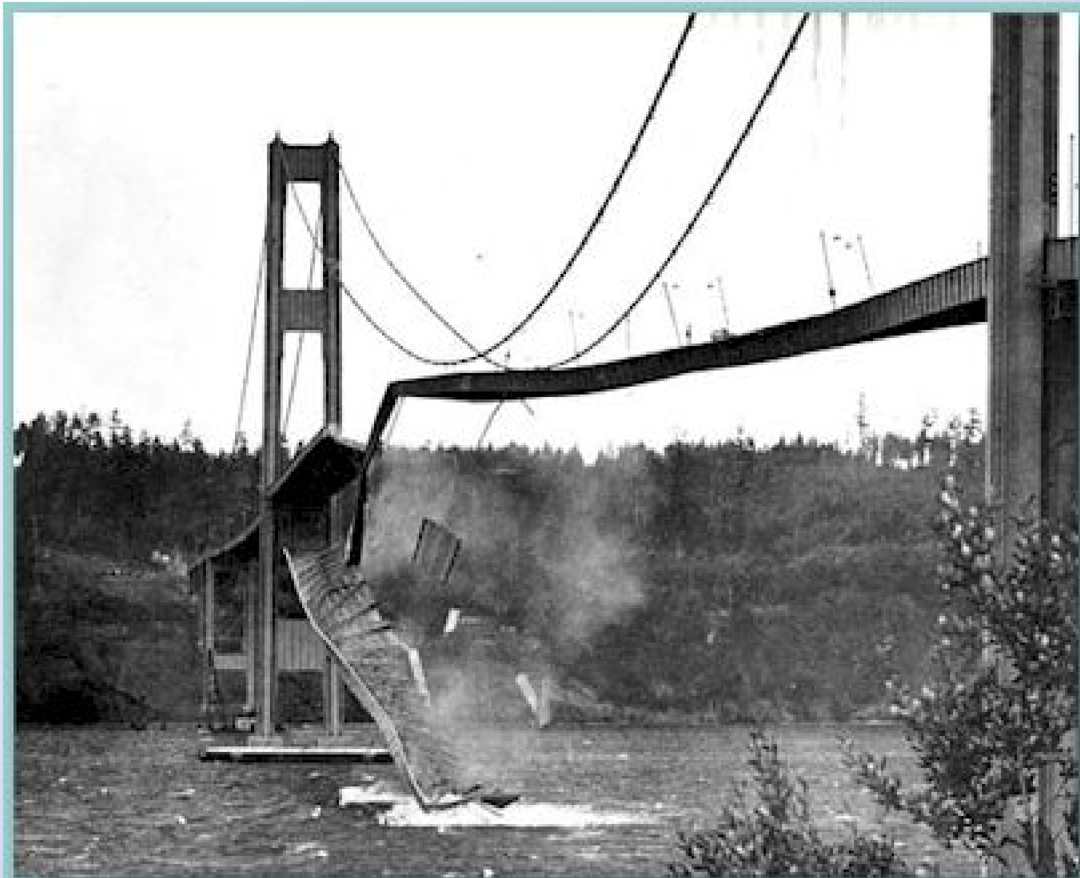
We chose the forcing term f(t) to be $\lambda sin\mu t$, a term that accurately models the frequency and amplitude of the oscillations observed before the collapse.

Equation 21 is the linear model for the bridge and equation 20 is the non-linear model.

The builders of the Tacoma Narrows Bridge should have used the non-linear equation, but due to their inability to solve it they were forced to use the linear.

This turned out to be a costly mistake. The linear equation gave them bad data as the model showed eventual torsional stability no matter what driving force and initial condition it was given.

Since the bridge actually acted according to the non-linear equation not the linear, the bridge did not settle down like the engineers of the bridge expected and the resulting torsional strain is believed to be a major contributor to the failure of the bridge.



Non-linear vs. Linear

First we will explore the long-term behavior of the linear model, with different initial conditions and driving terms of different periods (μ controls the period).

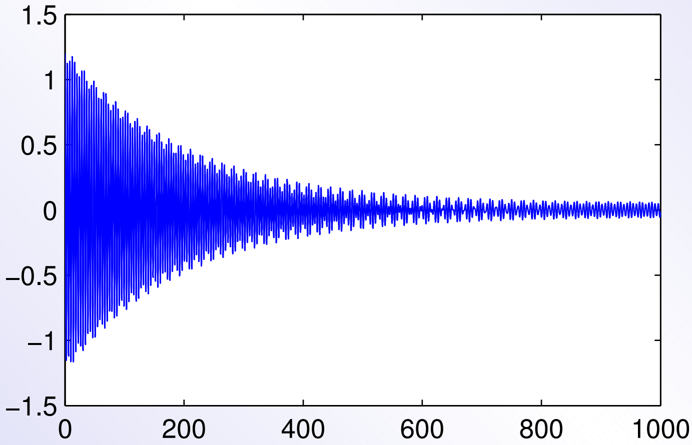


Figure 1a: Linear model with initial conditions $\theta = 1.2$, $\dot{\theta} = 0$, and $\lambda = .05$ (large push).
Note: vertical axis is theta in radians and horizontal is time for all graphs.

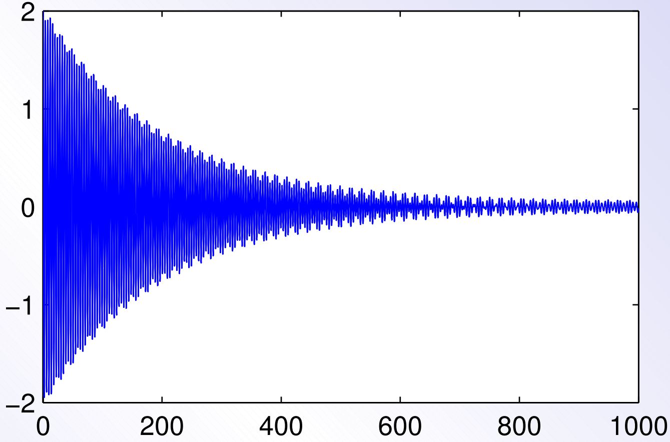


Figure 1b Linear model with initial conditions $\theta = 2$, $\dot{\theta} = 0$ and $\lambda = .05$ (very large push).

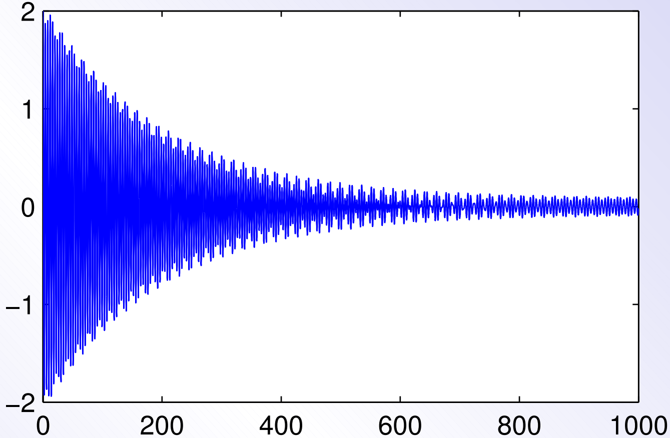


Figure 1c Linear model, $\theta = 2$, $\dot{\theta} = 0$ and $\lambda = .08$ (very large push and large driving term).

We can easily see that the linear model always settles down when given enough time.

Now let us explore the long-term behavior of the non-linear model. Here we see an interesting behavior, even if the period of the driving term is held constant the bridge can go crazy, all it needs is a large initial push. With sufficient push the torsional oscillations will never die down.

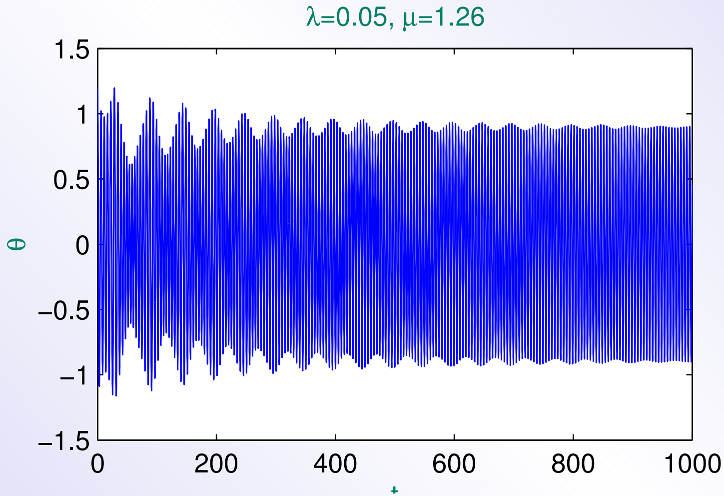


Figure 2a: non-linear model with a large initial push ($\lambda = .05$ and $\mu = 1.26$).

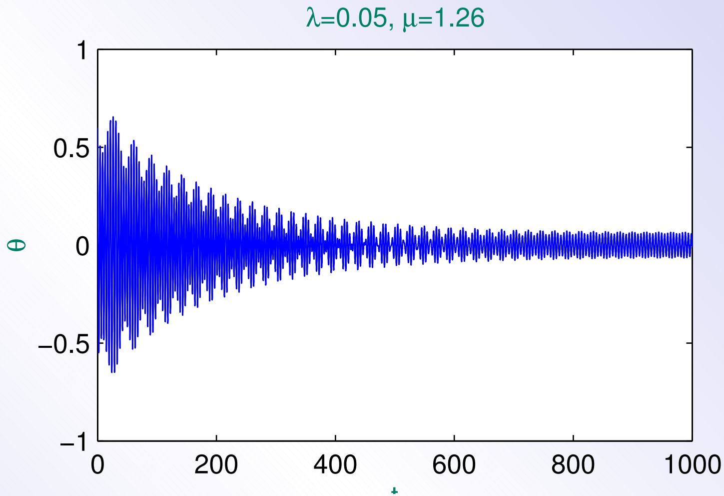


Figure 2b: non-linear model with a small initial push ($\lambda = .05$ and $\mu = 1.26$).

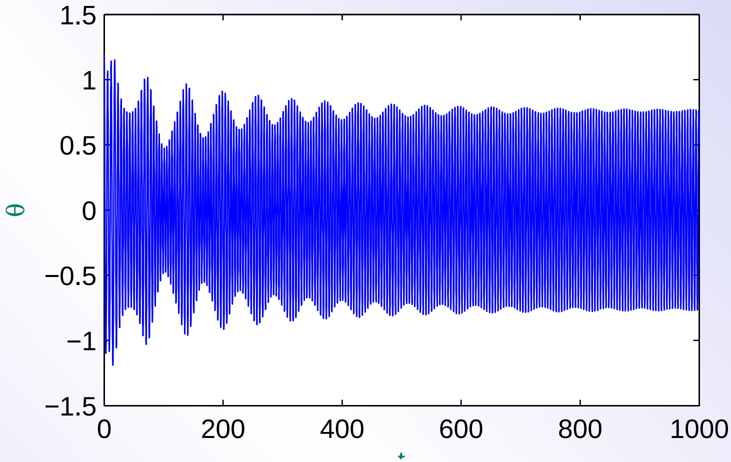


Figure 2c: non-linear model with a large initial push and a different frequency than in Figure 2a or 2b ($\lambda = .05$ and $\mu = 1.35$).

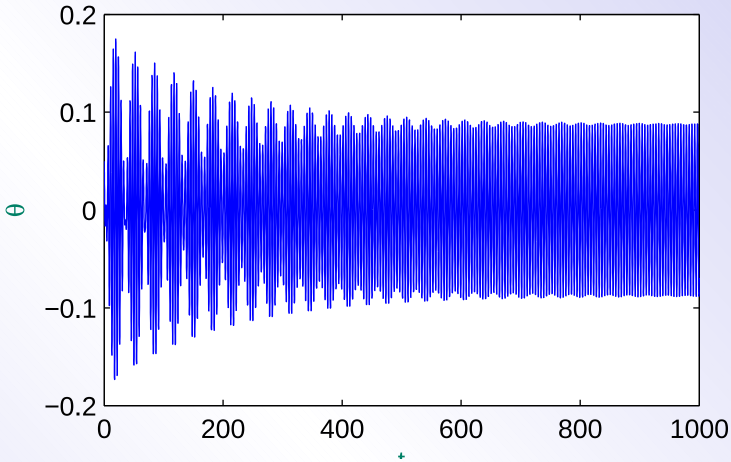


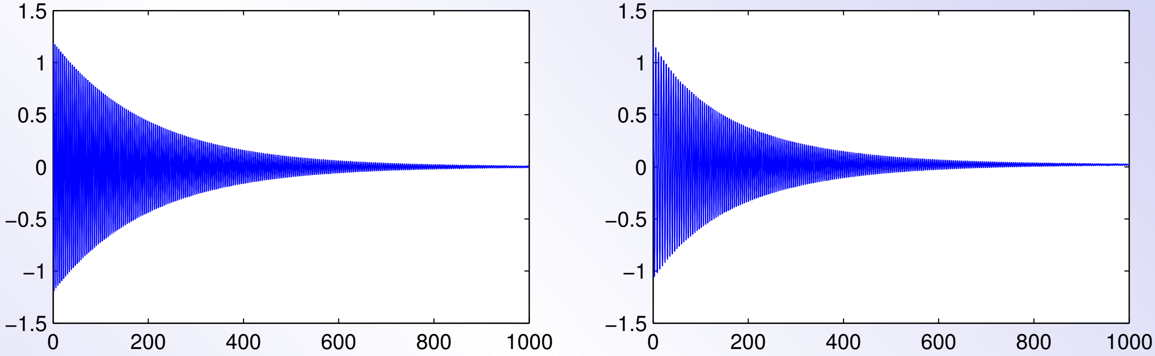
Figure 2d: non-linear model with a small initial push and the same frequency as Figure 2c ($\lambda = .05$ and $\mu = 1.35$).

This is an important fact to remember. Over a range of frequencies the non-linear model can exhibit behavior similar to the linear model or very different. It all depends on the initial push.

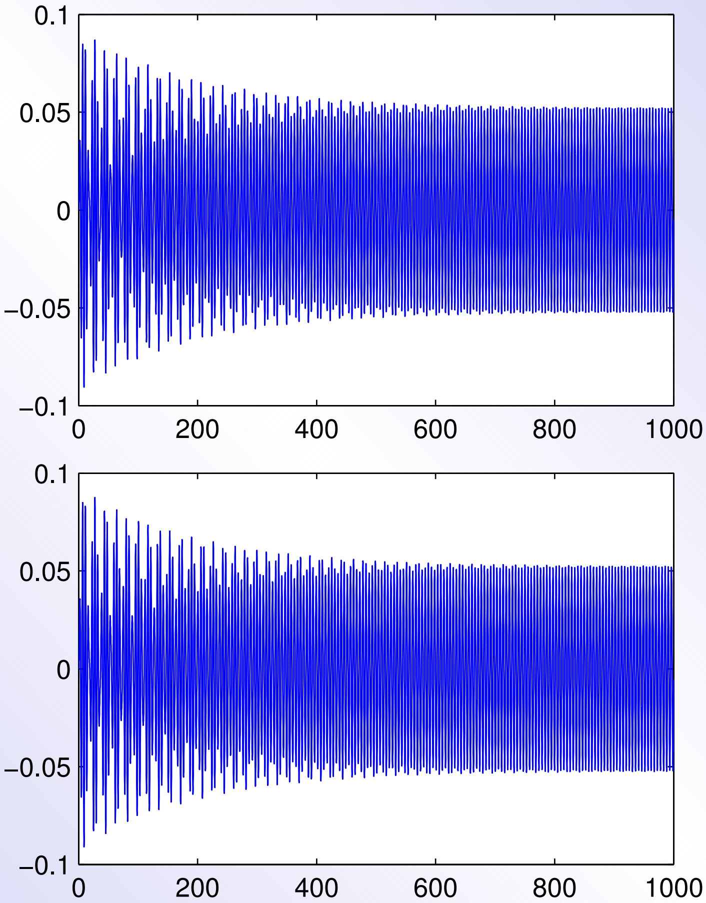
Applied to Tacoma Narrows

At the time of the bridge's construction, it was not possible to solve the non-linear model so the engineers chose to use the linear model. After seeing the behavior of the linear and non-linear model you can probably guess where we are going. We will start by doing a direct comparison of the linear and non-linear models. When given a large initial push and no driving terms both the models are very similar, so far so good.

The left graph is the linear model the right is the non-linear version.



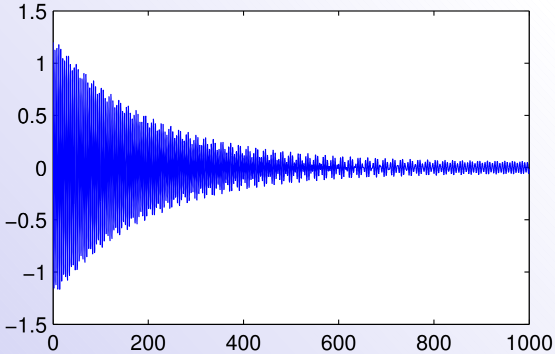
Now we start both models at equilibrium but add a small driving term ($\lambda = .05$). Once again both models appear very similar, and this is good.

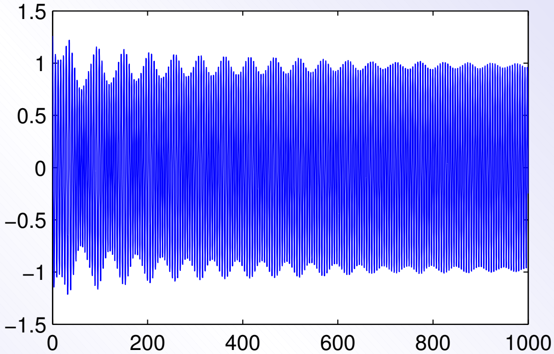


The top graph is the linear model the bottom is the non-linear version.

Finally we will look at both models starting with a large initial condition and a driving term. Here we see what you would probably expect, the linear model dies down, but the non-linear model does not. Large oscillations continue on for all time or until the eventual collapse of the bridge.

The left graph is the linear model the right is the non-linear version.





In Conclusion

Had the engineers of the time been able to solve the non-linear equation the error of using a incorrect linear model would not have been done. Using the correct non-linear model may have led to a more sturdy bridge that could still be standing today.