The Parachute Problem

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Differential Equations

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Objectives

- To present a basic model for the Parachute Problem as presented in many Differential Equation textbooks.
- Solve and explain problems with this basic model.
- To introduce an improved model that more accurately depicts a real life sky-diving jump.

Basic Model

Simple application of Newtonian Mechanics.

$$F = ma$$

$$F = F_g + F_d = ma$$

Where,

$$Fg = -mg$$

$$F_d = -kv$$

If deployment occurs at t_0 then,

$$k = \begin{cases} k_1, & 0 \le t < t_0 \\ k_2, & t \ge t_0 \end{cases}$$

The problem can be expressed as either a second-order differential equation (ODE) for position or as a first-order system of ODE's for the velocity and position. During the first interval the velocity satisfies the initial value problem

$$m\frac{dv}{dt} = -mg - k_1 v, \qquad v(0) = 0. \tag{1}$$

Solutions for velocity and position are well known.

$$v(t) = \frac{mg}{k_1}(e^{-k_1 t/m} - 1), \qquad 0 \le t < t_0$$

$$y(t) = y_0 - \frac{mg}{k_1}t - \frac{m^2 g}{k_1^2}(e^{-k_1 t/m} - 1), \qquad 0 \le t < t_0$$

During the second interval the velocity satisfies the I.V.P.

$$m\frac{dv}{dt} = -mg - k_2 v,$$

with the initial condition

$$v(t_0) = \frac{mg}{k_1}(e^{-k_1 t_0/m} - 1).$$

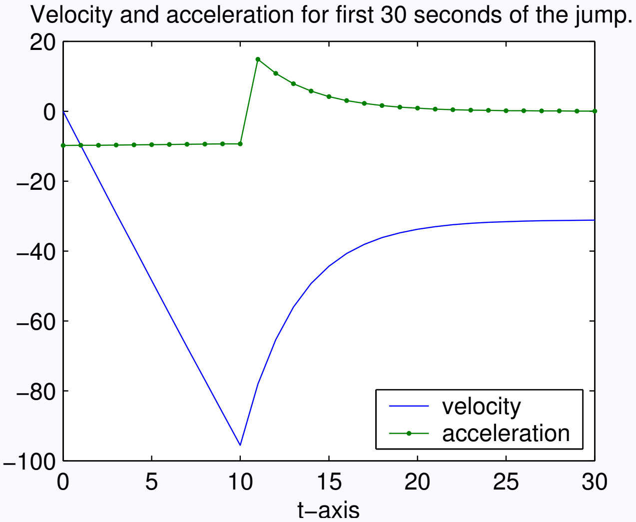
Therefore, the equation for velocity is

$$v(t) = \frac{mg}{k_2}(e^{-k_2(t-t_0)/m} - 1) + \frac{mg}{k_1}(e^{-k_2(t-t_0)/m})(e^{-k_1 t_0/m} - 1), \quad t \geq t_0.$$

Using a numerical solver we can plot these equations over time.

Results

Using MatLab we obtain the following graph,



Skydiving Physics

- Based on principles of Fluid Mechanics
- Relationship between viscous and inertial forces is described by the Reynolds Number.
- $Re = \rho dv/\mu$
- Ranges from 0 for a dust particle or larger objects in less viscous fluids to more than 10^8 for submarine in water.
- When $Re < 1$ viscous forces dominate and the drag is linear in velocity.
- When $Re > 10^3$ inertial forces dominate and the drag is quadratic in velocity.
- Therefore, Reynolds Numbers are essential to consider in the development of the model.

Calculation of Reynolds Number

- Assuming ρ an μ are constant at altitudes appropriate for skydiving. Where $\rho = 1$ kg/m^3 and μ=1.5x10^-5 Kg/m/s.
- Terminal velocity is a reasonable choice characteristic velocity when determining Reynolds Number.
- During free-fall $v \approx 45$ m/s≈ 100 mph.
- With chute deployed $v \approx 5.3$ m/s
- Thus, at each stage of the jump

$$Re > 10^6$$

- Therefore, drag is proportional to the square of velocity for our model.

Coefficient of Drag

- C_d is determined by the shape of the body and is found experimentally or through complex computational analysis.

$$F_d = \frac{1}{2}(C_d A \rho v^2)$$

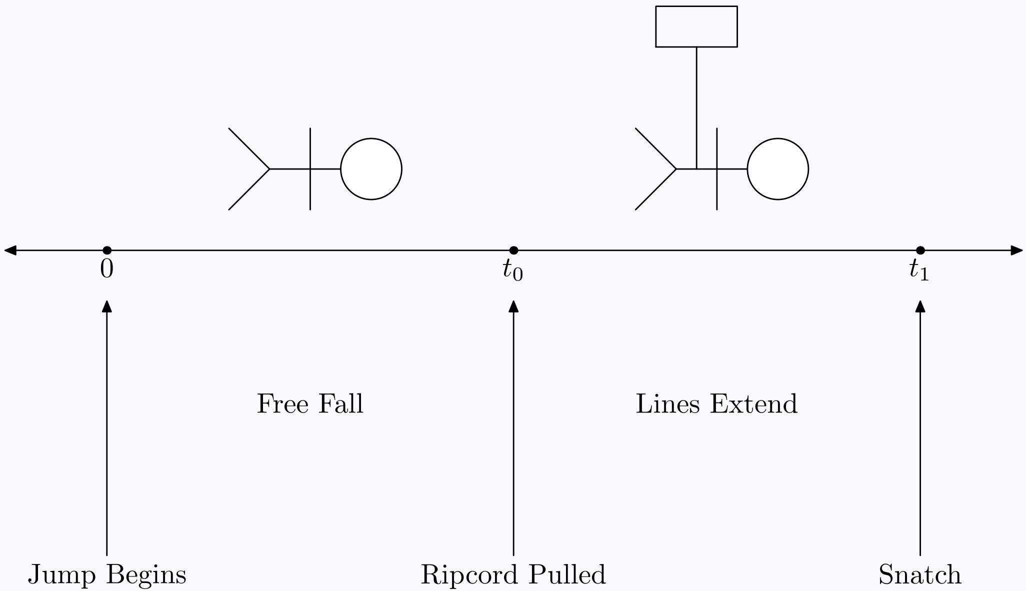
- Both the skydiver and their equipment generate separate drag forces, therefore,

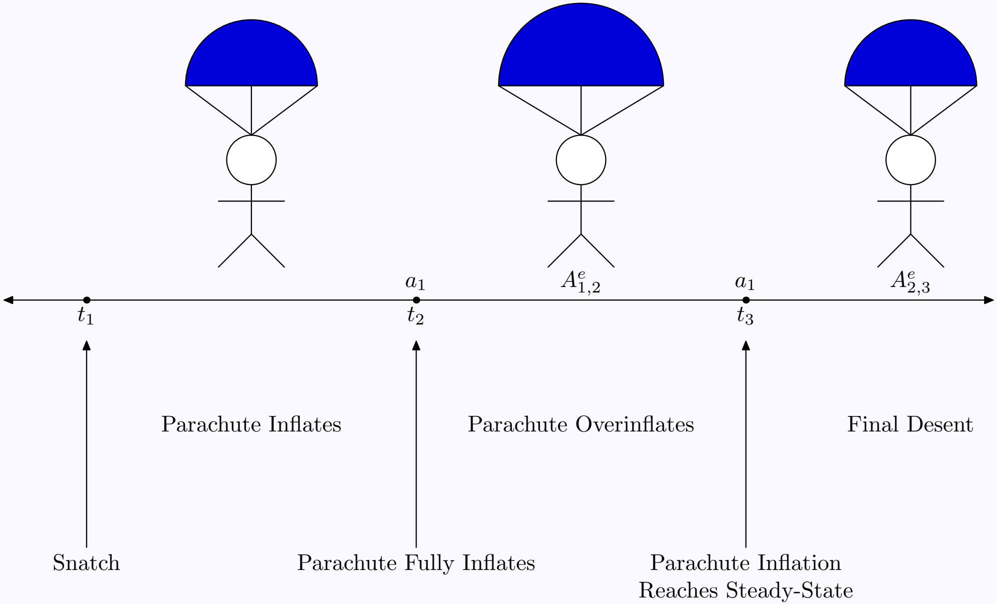
$$F_d = {F^b}_d + {F^e}_d = \frac{1}{2}\rho\,({C^b}_d A^b + {C^e}_d A^e)v^2$$

Shape	Reynolds Number	C_d
Hemispherical Shell	$Re > 10^3$	1.33
Flat Strip	$Re > 10^3$	1.95
Cylinder	$Re > 5x10^5$	≈ 0.35

- During free-fall the skydiver is in a horizontal position and can be represented by a flat strip with $A \approx 0.5$ m^2.
- After parachute deployment the skydiver is in a vertical position and can be represented by a long cylinder with $A \approx 0.1$ m^2.
- The canopy can be represented by a hemispherical shell where $A \approx 43.8$ m^2.

Time-Line of Jump




a_1
$A^e_{1,2}$
a_1
$A^e_{2,3}$
t_1
t_2
t_3
Parachute Inflates
Parachute Overinflates
Final Desent
Snatch
Parachute Fully Inflates
Parachute Inflation
Reaches Steady-State

Area of Skydiver's Body

a_1	b_0	b_1	h	l
$43.8m^2$	$0.5m^2$	$0.1m^2$	$1.78m$	$8.96m$

m	t_0	t_1	t_2	t_3
$97.2kg$	$10s$	$10.5s$	$11.5s$	13.2

$$A^b(t) = \begin{cases} b_0, & t \leq t_0 \\ b_0, & t_0 < t \leq t_1 \\ b_1, & t_1 < t \leq t_2 \\ b_1, & t_2 < t \leq t_3 \\ b_1, & t \geq t_3 \end{cases}$$

Coefficient of Drag for Skydiver

a_1	b_0	b_1	h	l
$43.8m^2$	$0.5m^2$	$0.1m^2$	$1.78m$	$8.96m$

m	t_0	t_1	t_2	t_3
$97.2kg$	$10s$	$10.5s$	$11.5s$	13.2

$$C_d{}^b(t) = \begin{cases} 1.95, & t \le t_0 \\ 1.95, & t_0 < t \le t_1 \\ 0.35h, & t_1 < t \le t_2 \\ 0.35h, & t_2 < t \le t_3 \\ 0.35h, & t \ge t_3 \end{cases}$$

Area of the Equipment

a_1	b_0	b_1	h	l
$43.8m^2$	$0.5m^2$	$0.1m^2$	$1.78m$	$8.96m$

m	t_0	t_1	t_2	t_3
$97.2kg$	$10s$	$10.5s$	$11.5s$	13.2

$$A^e(t) = \begin{cases} 0.0, & t \le t_0 \\ b_1, & t_0 < t \le t_1 \\ {A^e}_{1,2}(t), & t_1 < t \le t_2 \\ {A^e}_{2,3}(t), & t_2 < t \le t_3 \\ a_1, & t \ge t_3 \end{cases}$$

Coefficient of Drag for Equipment

a_1	b_0	b_1	h	l
$43.8m^2$	$0.5m^2$	$0.1m^2$	$1.78m$	$8.96m$

m	t_0	t_1	t_2	t_3
$97.2kg$	$10s$	$10.5s$	$11.5s$	13.2

$$C_d{}^e(t) = \begin{cases} 0.0, & t \leq t_0 \\ 0.35l\frac{t-t_0}{t_1-t_0}, & t_0 < t \leq t_1 \\ 1.33, & t_1 < t \leq t_2 \\ 1.33, & t_2 < t \leq t_3 \\ 1.33, & t \geq t_3 \end{cases}$$

Improved Model

$$m\frac{dv}{dt} = -mg + kv^2, \qquad v(0) = 0$$

where

$$k = 1/2\rho({C^b}_d A^b + {C^e}_d A^e).$$

Thus,

$$k = \frac{1}{2}\rho \begin{cases} 1.95b_0, & t \le t_0 \\ 1.95b_0 + 0.35b_1 l \frac{t-t_0}{t_1-t_0}, & t_0 < t \le t_1 \\ 0.35b_1 h + 1.33{A^e}_{1,2}(t), & t_1 < t \le t_2 \\ 0.35b_1 h + 1.33{A^e}_{2,3}(t), & t_2 < t \le t_3 \\ 0.35b_1 h + 1.33a_1, & t \ge t_3 \end{cases}$$

Continuity at End Points

- At t_0,

$$1.95b_0 = 1.95b_0 + 0.35b_1 l\left(\frac{t - t_0}{t_1 - t_0}\right)$$

Substituting t_0 for t;

$$1.95b_0 = 1.95b_0 + 0.35b_1 l\left(\frac{t_0 - t_0}{t_1 - t_0}\right)$$

$$1.95b_0 = 1.95b_0$$

- At t_1,

$$
\begin{aligned}
1.95b_0 + 0.35b_1 l &= 0.35b_1 h + 1.33{A^e}_{1,2}(t_1) \\
1.95b_0 + 0.35b_1 l - 0.35b_1 h &= 1.33{A^e}_{1,2}(t_1) \\
1.95b_0 + 0.35b_1(l - h) &= 1.33{A^e}_{1,2}(t_1) \\
{A^e}_{1,2}(t_1) &= \frac{1.95b_0 + 0.35b_1(l - h)}{1.33}
\end{aligned}
$$

- At t_2,

$$0.35 b_1 h + 1.33 {A^e}_{1,2}(t_2) = 0.35 b_1 h + 1.33 {A^e}_{2,3}(t_2)$$
$${A^e}_{1,2}(t_2) = {A^e}_{2,3}(t_2)$$

- At t_3,

$$0.35 b_1 h + 1.33 {A^e}_{2,3}(t_3) = 0.35 b_1 h + 1.33 a_1$$
$${A^e}_{2,3}(t_3) = a_1$$

Equation for ${A^e}_{1,2}(t)$

$$ {A^e}_{1,2}(t) = \alpha_0 e^{\beta_0 (t-t_1)/(t_2-t_1)} $$

where , $\alpha_0 = \dfrac{1.95b_0 + 0.35b_1(l-h)}{1.33}$ and , $\beta_0 = \ln\left(\dfrac{a_1}{\alpha_0}\right)$.

Note:

$$ {A^e}_{1,2}(t_1) = \alpha_0 e^{\beta_0 (t_1-t_1)/(t_2-t_1)} $$

$$ {A^e}_{1,2}(t_1) = \alpha_0. $$

And,

$$ {A^e}_{1,2}(t_2) = \alpha_0 e^{\beta_0 (t_2-t_1)/(t_2-t_1)} $$

$$ {A^e}_{1,2}(t_2) = \alpha_0 e^{\ln(a_1/\alpha_0)} $$

$$ {A^e}_{1,2}(t_2) = a_1. $$

Equation for ${A^e}_{2,3}(t)$

$$A^e_{2,3}(t) = a_1 \left(1 + \beta_1 \sin\left(\pi \frac{t - t_2}{t_3 - t_2}\right)\right)$$

where , $\beta_1 = 0.15.$

Note:

$$A^e_{2,3}(t_2) = a_1 \left(1 + \beta_1 \sin\left(\pi \frac{t_2 - t_2}{t_3 - t_2}\right)\right)$$

$$A^e_{2,3}(t_2) = a_1(1)$$

$$A^e_{2,3}(t_2) = a_1.$$

And,

$$A^e_{2,3}(t_3) = a_1 \left(1 + \beta_1 \sin\left(\pi \frac{t_3 - t_2}{t_3 - t_2}\right)\right)$$

$$A^e_{2,3}(t_3) = a_1.$$

Results