Pursuit Curves

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Abstract

This paper will discuss the differential equations which describe curves of pure pursuit, in which the pursuer's velocity vector is always pointing directly toward the pursued.

1. Pirates!

We will begin with what is generally assumed to be the original pursuit problem, posed by French mathematician and hydrographer Pierre Bouger in 1732. The problem was published in the French Academy's *Mémoires de l'Académie Royale des Sciences* and it deals with a pirate ship pursuing a merchant vessel. The merchant vessel is taken to be at $(x_0, 0)$ at time $t = 0$, and travels at a constant speed V_m along the vertical line $x = x_0$. (See Figure 1). The pirate ship starts at $(0, 0)$ at time $t = 0$ and travels at constant speed V_p along a curved path, such that its velocity vector is always pointing directly at the merchant vessel. This is what is defined as a pure pursuit, and is described geometrically in Figure 1. The problem is to determine the equation $y = y(x)$ for the pirate's curve of pursuit.

If the location of the pirate ship at time $t \geq 0$ is the point (x, y), and the merchant vessel's position is the point $(x_0, V_m t)$, the slope of the tangent line to the pursuit curve at (x, y) is

$$\frac{dy}{dx} = \frac{V_m t - y}{x_0 - x} = \frac{y - V_m t}{x - x_0}\ .$$

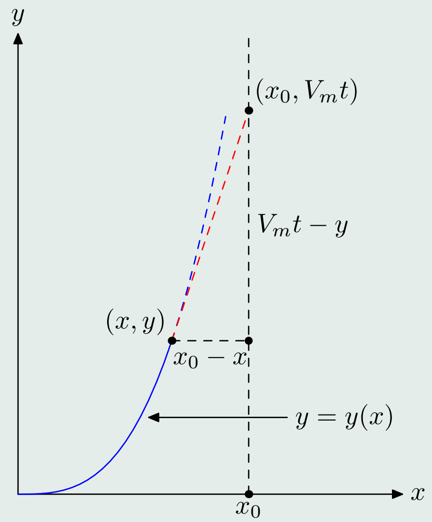


Figure 1: The geometry of the pursuit problem.

Solving this for t,

$$t = \frac{y}{V_m} - \frac{(x - x_0)}{V_m} \frac{dy}{dx} . \tag{1}$$

Further, we know that the pirate ship has always sailed a distance of $V_p t$ along the pursuit curve. Employing the arc length formula, we get

$$V_p t = \int_0^x \sqrt{1 + \left(\frac{dy}{dz}\right)^2} dz .$$

Solving this equation for t,

$$t = \frac{1}{V_p} \int_0^x \sqrt{1 + \left(\frac{dy}{dz}\right)^2} dz . \tag{2}$$

From equations (1) and (2), we get

$$\frac{1}{V_p} \int_0^x \sqrt{1 + \left(\frac{dy}{dz}\right)^2} dz = \frac{y}{V_m} - \frac{(x - x_0)}{V_m} \frac{dy}{dx} ,$$

which, letting $p(x) = dy/dx$, becomes

$$\frac{1}{V_p} \int_0^x \sqrt{1 + [p(z)]^2} dz = \frac{y}{V_m} - \frac{(x - x_0)}{V_m} p(x) .$$

Differentiating both sides gives

$$\frac{1}{V_p} \sqrt{1 + [p(x)]^2} = \frac{1}{V_m} \frac{dy}{dx} - \frac{(x - x_0)}{V_m} \frac{dp}{dx} - \frac{1}{V_m} p(x) ,$$

which, after replacing dy/dx on the right hand side with $p(x)$, reduces to

$$(x - x_0) \frac{dp}{dx} = -n\sqrt{1 + [p(x)]^2} , \tag{3}$$

where $n = V_m/V_p$.

This differential equation is solvable analytically. The first step is to separate the variables, letting $p = p(x)$.

$$\frac{dp}{\sqrt{1+p^2}} = \frac{-ndx}{x - x_0} \ .$$

We then integrate both sides using a table of integrals [3],

$$\ln(p + \sqrt{1+p^2}) + C = -n \ln(x_0 - x) \ . \tag{4}$$

Now, since $p = dy/dx$, and the pirate's velocity is always pointed directly toward the merchant ship, at time $t = 0$ the pirate's velocity is $dy/dx = 0$. Solving for C, we find that $C = -n \ln(x_0)$. Inserting this result into equation (4),

$$\ln(p + \sqrt{1+p^2}) - n \ln(x_0) = -n \ln(x_0 - x)$$
$$\ln(p + \sqrt{1+p^2}) + \ln(x_0 - x)^n - \ln(x_0)^n = 0 \ .$$

Applying the properties of logarithms from algebra,

$$0 = \ln(p + \sqrt{1+p^2}) + \ln\left(\frac{x_0 - x}{x_0}\right)^n$$
$$0 = \ln\left[(p + \sqrt{1+p^2})\left(1 - \frac{x}{x_0}\right)^n\right] \ . \tag{5}$$

We know from algebra that $\ln(1) = 0$, so

$$(p + \sqrt{1+p^2})\left(1 - \frac{x}{x_0}\right)^n = 1 \ .$$

We want to solve for p, so we divide:

$$p + \sqrt{1+p^2} = \left(1 - \frac{x}{x_0}\right)^{-n} \ .$$

At this point, a new variable, q, will be introduced, where $q = (1 - x/x_0)^{-n}$. This will simplify the algebra of the next few steps.

$$\begin{aligned} p + \sqrt{1+p^2} &= q \\ 1 + p^2 &= (q-p)^2 \\ 1 + p^2 &= q^2 - 2qp + p^2 \\ 2qp &= q^2 - 1 \end{aligned}$$

Now, solving for p gives us

$$p = \frac{1}{2}\left(q - \frac{1}{q}\right) . \qquad (6)$$

Substituting back in for p and q, we have

$$\frac{dy}{dx} = \frac{1}{2}\left[\left(1 - \frac{x}{x_0}\right)^{-n} - \left(1 - \frac{x}{x_0}\right)^{n}\right] .$$

We can now integrate both sides with respect to x:

$$y(x) + C = \frac{1}{2}\int \left(1 - \frac{x}{x_0}\right)^{-n} dx - \frac{1}{2}\int \left(1 - \frac{x}{x_0}\right)^{n} dx .$$

Letting $u = 1 - x/x_0$, $du = (-1/x_0)\,dx$, our integral becomes

$$y(x) + C = \frac{1}{2}\int -x_0 u^{-n} du - \frac{1}{2}\int -x_0 u^{n} du .$$

We are now ready to integrate the right hand side, which yields the following result

$$y(x) + C = -\frac{1}{2}x_0\left(\frac{u^{-n+1}}{-n+1}\right) + \frac{1}{2}x_0\left(\frac{u^{n+1}}{n+1}\right) .$$

Simplifying,

$$\begin{aligned} y(x) + C &= \frac{1}{2} x_0 \left[\frac{u^{n+1}}{n+1} - \frac{u^{-n+1}}{-n+1} \right] \\ &= \frac{1}{2} x_0 u \left[\frac{u^{n}}{n+1} - \frac{u^{-n}}{-n+1} \right] . \end{aligned}$$

Now we can substitute back in for u,

$$\begin{aligned} y(x) + C &= \frac{1}{2} x_0 \left(1 - \frac{x}{x_0} \right) \left[\frac{(1 - x/x_0)^n}{1+n} - \frac{(1 - x/x_0)^{-n}}{1-n} \right] \\ &= \frac{1}{2} (x_0 - x) \left[\frac{(1 - x/x_0)^n}{1+n} - \frac{(1 - x/x_0)^{-n}}{1-n} \right] . \end{aligned}$$

We know that the pirate ship starts at the origin, that is, $y(0) = 0$. Using this fact, we can solve for C:

$$\begin{aligned} C &= \frac{1}{2} x_0 \left(\frac{1}{1+n} - \frac{1}{1-n} \right) \\ &= \frac{1}{2} x_0 \left[\frac{-2n}{1-n^2} \right] \\ C &= -\frac{n}{1-n^2} x_0 . \end{aligned}$$

We now have our solution to equation (3),

$$y(x) = \frac{1}{2} (x_0 - x) \left[\frac{(1 - x/x_0)^n}{1+n} - \frac{(1 - x/x_0)^{-n}}{1-n} \right] + \frac{n}{1-n^2} x_0 , \tag{7}$$

where $n = V_m/V_p$.

1.1. Matlab

We can check our solution in Matlab by comparing it's graph with that of equation (3), making use of Matlab's numerical solver, `ode45`.

First, we will substitute dy/dx back in for $p(x)$ in equation (3), noting that $dp/dx = d^2y/dx^2$.

$$\frac{d^2y}{dx^2} = \frac{-n\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{x - x_0}$$

Now, the `ode45` solver cannot handle second-order differential equations such as ours, but we can convert the second-order differential equation into a system of two first-order differential equations, and feed them to `ode45` as a vector.

First, we let

$$\begin{aligned} u_1 &= y \\ u_2 &= y' \ . \end{aligned}$$

Next, differentiate,

$$\begin{aligned} u_1' &= y' \\ u_2' &= y'' \ . \end{aligned}$$

Now we can substitute appropriately, making sure our new equations contain only u's and no y's,

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= \frac{-n\sqrt{1+u_2^2}}{x - x0} \ . \end{aligned}$$

We define the vector $\mathbf{u}$ as follows.

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\mathbf{u}' = \begin{bmatrix} u_1' \\ u_2' \end{bmatrix}$$

We see that $\mathbf{u}$ is a column vector, with u_1 as its first entry and u_2 as its second. We will take advantage of this to write a function m-file for our system using the following code:

```
function uprime = piratechase(x,u,n,x0)
uprime = zeros(2,1);
uprime(1) = u(2);
uprime(2) = -n*sqrt(1+(u(2))^2)/(x-x0);
```

Then, we can write a script file that calls on `ode45` to solve the system numerically. We begin by choosing a time span, initial conditions, and values for n and x_0.

```
tspan=[0,15];
init=[0,0];
n=.3;
x0=10;
```

We then "capture" the results from `ode45` in the variables `t` and `u`, and plot the variable `t` versus the first `u`-column, which represents u_1, or y. This results in the curve in

```
[t,u] = ode45(@piratechase,tspan,init,[],n,x0);
plot(t,u(:,1));
```

The merchant's path is plotted with the following code.

```
y=linspace(0,4);
x1=x0*ones(size(y));
plot(x1,y,'r--')
```

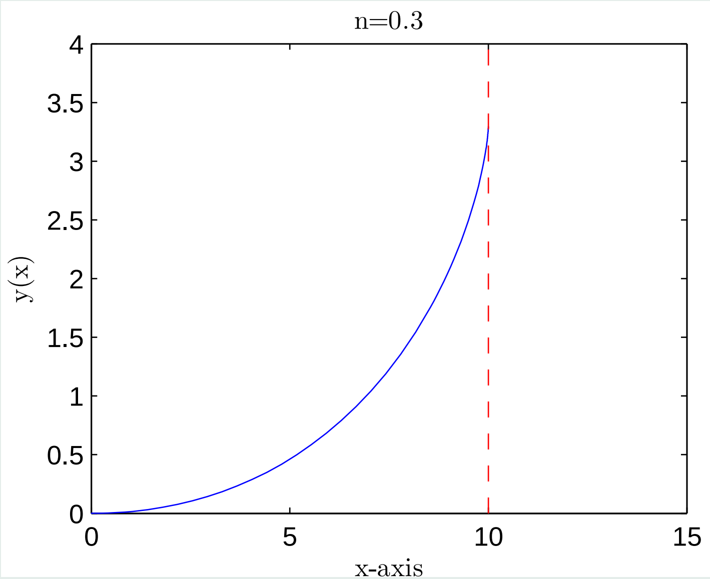


Figure 2: Pirate's path using ode45.

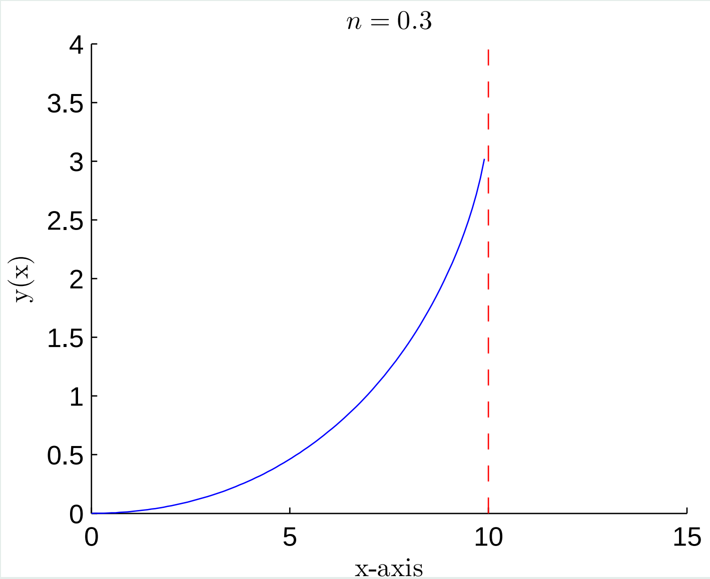


Figure 3: Graph of the solution $y(x)$.

The above code produces the plot seen in figure 2.

Next we will graph the solution we found in the previous section. This is done by first defining n and x_0,

```
n=.3; x0=10;\ ,
```

then we define a domain for x, our equation for $y(x)$, and plot the result with a `line` command.

```
x=linspace(0,x0);
y = .5*(x0-x).*((1-x/x0).^n/(1+n)-(1-x/x0).^-n/(1-n))+n*x0/(1-n^2);
line(x,y)
```

This code produces the plot in figure 3. We can see that the two graphs are identical, so our solution is confirmed.

2. Pursuit Curve for a Circle

The problem posed by A.S. Hathaway, as stated by Nahin (2007), is this: "A dog at the center of a circular pond makes straight for a duck which is swimming [counterclockwise] along the edge of the pond. If the rate of swimming of the dog is to the rate of swimming of the duck as $n:1$, determine the equation of the curve of pursuit..."[1].

We will begin with a vector approach, illustrated by figure 4 (since *dog* and *duck* both start with d, we will use $\mathbf{h}(t)$ to represent the *hound's* position vector). The vector $\boldsymbol{\rho}(t)$ represents the distance between the hound and the duck, so we have

$$\mathbf{d}(t) = \mathbf{h}(t) + \boldsymbol{\rho}(t) \ .$$

We can write this in Cartesian coordinates (in the complex plane) as

$$\mathbf{d}(t) = x_d(t) + iy_d(t) \tag{8}$$

and

$$\mathbf{h}(t) = x_h(t) + iy_h(t) \ . \tag{9}$$

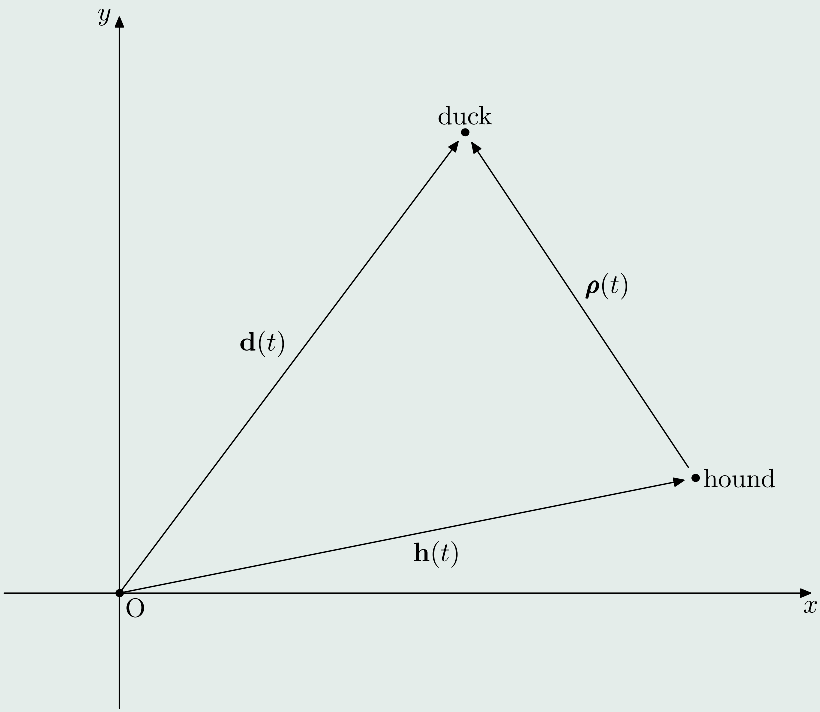


Figure 4: A vector approach.

Since the hound's velocity vector is always pointing directly toward the duck (the definition of a pure pursuit), $d\mathbf{h}(t)/dt$ is always pointing from hound to duck, and, $|d\mathbf{h}(t)/dt|$ is the hound's speed. Further, since $\boldsymbol{\rho}(t) = \mathbf{d}(t) - \mathbf{h}(t)$ points from the hound to the duck, a unit vector pointing from the hound to the duck is

$$\frac{\boldsymbol{\rho}(t)}{|\boldsymbol{\rho}(t)|} \ ,$$

so the hound's velocity vector is

$$\frac{d\mathbf{h}(t)}{dt} = \left|\frac{d\mathbf{h}(t)}{dt}\right| \cdot \frac{\boldsymbol{\rho}(t)}{|\boldsymbol{\rho}(t)|} \ . \tag{10}$$

The velocity vector of the duck (which is known by the statement of the problem) is given by

$$\frac{d\mathbf{d}(t)}{dt} = \frac{dx_d}{dt} + i\frac{dy_d}{dt} \ ,$$

so the duck's speed is

$$\left|\frac{d\mathbf{d}(t)}{dt}\right| = \sqrt{\left(\frac{dx_d}{dt}\right)^2 + \left(\frac{dy_d}{dt}\right)^2} \ .$$

We know that the speed of the dog is n times that of the duck, that is

$$\left|\frac{d\mathbf{h}(t)}{dt}\right| = n\sqrt{\left(\frac{dx_d}{dt}\right)^2 + \left(\frac{dy_d}{dt}\right)^2} \ ,$$

so equation (10) becomes

$$\frac{d\mathbf{h}(t)}{dt} = n\sqrt{\left(\frac{dx_d}{dt}\right)^2 + \left(\frac{dy_d}{dt}\right)^2} \cdot \frac{\mathbf{d}(t) - \mathbf{h}(t)}{|\mathbf{d}(t) - \mathbf{h}(t)|} \ .$$

This can be written in Cartesian coordinates as

$$\frac{dx_h}{dt} + i\frac{dy_h}{dt} = n\sqrt{\left(\frac{dx_d}{dt}\right)^2 \left(\frac{dy_d}{dt}\right)^2} \cdot \frac{(x_d - x_h) + i(y_d - y_h)}{\sqrt{(x_d - x_h)^2 + (y_d - y_h)^2}} \ . \tag{11}$$

Equating the real and imaginary parts of (11) gives the general differential equations of pursuit. Note that these equations work for any path followed by the prey that can be written parametrically.

$$\frac{dx_h}{dt} = n\sqrt{\left(\frac{dx_d}{dt}\right)^2 + \left(\frac{dy_d}{dt}\right)^2} \cdot \frac{x_d - x_h}{\sqrt{(x_d - x_h)^2 + (y_d - y_h)}} \tag{12}$$

and

$$\frac{dy_h}{dt} = n\sqrt{\left(\frac{dx_d}{dt}\right)^2 + \left(\frac{dy_d}{dt}\right)^2} \cdot \frac{y_d - y_h}{\sqrt{(x_d - x_h)^2 + (y_d - y_h)}} \ . \tag{13}$$

If the duck swims around the unit circle, starting at $(1, 0)$, then $x_d = \cos(t)$ and $y_d = \sin(t)$, so the first term in equations (12) and (13) becomes

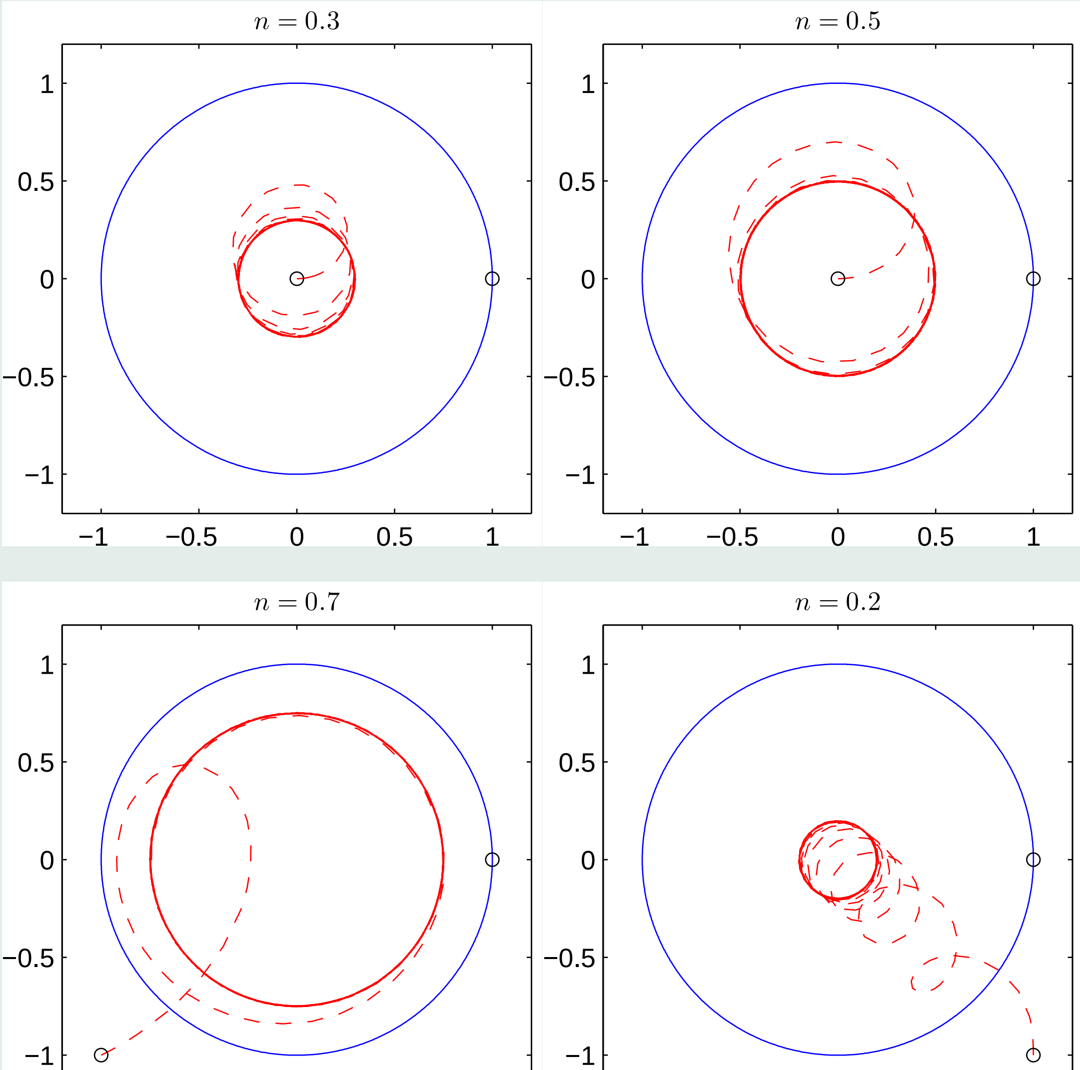
$$n\sqrt{\left(\frac{dx_d}{dt}\right)^2 + \left(\frac{dy_d}{dt}\right)^2} = n\sqrt{\sin^2(t) + \cos^2(t)} = n \ .$$

When we substitute this in equations (12) and (13), we get the differential equations of pursuit when the prey is following the unit circle:

$$\frac{dx_h}{dt} = n\frac{\cos(t) - x_h}{\sqrt{(\cos(t) - x_h)^2 + (\sin(t) - y_h)^2}} \tag{14}$$

and

$$\frac{dy_h}{dt} = n\frac{\sin(t) - y_h}{\sqrt{(\cos(t) - x_h)^2 + (\sin(t) - y_h)^2}} \ . \tag{15}$$



$n = 0.3$
$n = 0.5$
$n = 0.7$
$n = 0.2$
1
0.5
0
−0.5
−1
−0.5
0
0.5
1

We can graph these equations very easily in Matlab, as we did above. The following code produced to plots in figure (5). As before, we begin by declaring the time span and initial conditions, as well as n, where the hound moves n times the speed of the duck.

```
tspan = [0,100];
init = [0;0];
n = 0.3;
```

The various starting points for the hound can be achieved by varying the `init` vector. For instance, the graph where $n = 0.7$ was made with the command `init = [-1;-1]`. Next we call on `ode45`, storing the results in the variables `t` and `y`, then plot y_1 versus y_2,

```
[t,y]=ode45(@pursuit,tspan,init,[],n);
plot(y(:,1),y(:,2),'r--')
```

After obtaining numerical results for several values of n and various initial positions of the hound, it appears that the hound eventually reaches what looks like a circular limit cycle, the radius of which seems to be dependent on n. We can investigate this behavior more closely by switching coordinate systems. We will do so with the following approach.

3. Alternate Equations

We will begin by taking the duck's path to be a circle of radius a, and the duck's initial position, at time $t = 0$ at the point $(a, 0)$, as seen in Figure 6. At time $t > 0$ the duck has moved through an arc angle θ, (a distance $a\theta$), so the dog (swimming n times faster than the duck), moves a distance of $s = an\theta$ to the point (x, y). The dog is executing a pure pursuit, so the tangent line to the dog's pursuit curve at the point (x, y) will pass through the duck's position. This tangent line makes an angle ω with the x-axis, and the distance between the dog and the duck (along the tangent line) is ρ.

The slope of the tangent line at the point (x, y) is $\tan(\omega)$, so the equation of the tangent line, in slope-intercept form, is given by

$$y = x\tan(\omega) + b = x\frac{\sin(\omega)}{\cos(\omega)} + b\ . \tag{16}$$

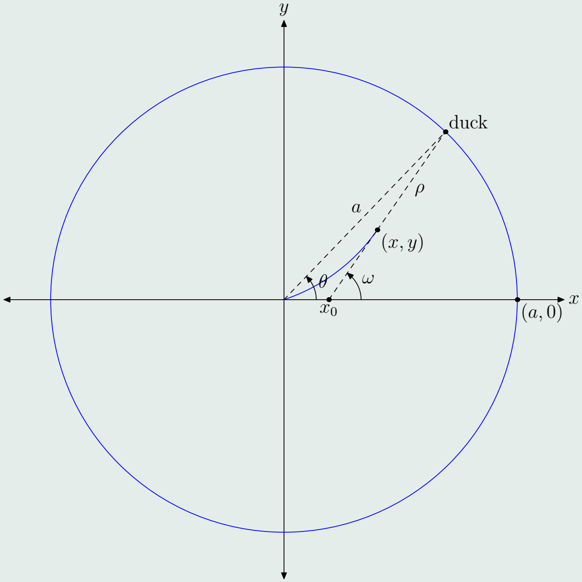


Figure 6: The geometry of Hathaway's problem.

To find b, we use the fact that the line passes through the duck's position at the point $(a\cos(\theta), a\sin(\theta))$:

$$a\sin(\theta) = a\cos(\theta)\frac{\sin(\omega)}{\cos(\omega)} + b\ .$$

To clear fractions, we multiply both sides by $\cos(\omega)$,

$$a\sin(\theta)\cos(\omega) = a\cos(\theta)\sin(\omega) + b\cos(\omega)$$

then, solve for b.

$$\begin{aligned} b\cos(\omega) &= a\sin(\theta)\cos(\omega) - a\cos(\theta)\sin(\omega) \\ &= a\sin(\theta-\omega) \\ &= -a\sin(\omega-\theta) \end{aligned}$$

From here, we multiply equation (16) by $\cos(\omega)$ to get

$$y\cos(\omega) - x\sin(\omega) = -a\sin(\omega-\theta)\ . \tag{17}$$

Now we must find the equation of the normal line to the tangent line, that passes through the point (x, y). We know that the slope of the normal line will be $-1/\tan(\omega)$, so the equation of the normal line has the form

$$y = -x\frac{cos(\omega)}{sin(\omega)} + b\ . \tag{18}$$

Again, to find b we need the coordinates of another point through which the normal line passes.

From Figure 7 we can see that the radius from the center of the pond to the duck's position is $a = d_1 + d_2$, and

$$d_2 = \frac{\rho}{\cos(\omega-\theta)}\ .$$

This leads to

$$d_1 = a - \frac{\rho}{\cos(\omega - \theta)} = \frac{a\cos(\omega - \theta) - \rho}{\cos(\omega - \theta)} \ ,$$

which tells us that the x and y values of the point of intersection are

$$x = \cos(\theta)\frac{a\cos(\omega - \theta) - \rho}{\cos(\omega - \theta)} \ , \ y = \sin(\theta)\frac{a\cos(\omega - \theta) - \rho}{\cos(\omega - \theta)} \ .$$

Substituting these values for x and y in equation (18) gives us

$$\sin(\theta)\frac{a\cos(\omega - \theta) - \rho}{\cos(\omega - \theta)} = -\cos(\theta)\frac{a\cos(\omega - \theta) - \rho}{\cos(\omega - \theta)} \cdot \frac{\cos(\omega)}{\sin(\omega)} + b \ ,$$

or

$$b = \frac{[a\cos(\omega - \theta) - \rho][\sin(\theta)\sin(\omega) + \cos(\theta)\cos(\omega)]}{\sin(\omega)\cos(\omega - \theta)} \ .$$

Here we note that $\sin(\theta)\sin(\omega) + \cos(\theta)\cos(\omega) = \cos(\omega - \theta)$, so

$$b = \frac{a\cos(\omega - \theta) - \rho}{\sin(\omega)} \ ,$$

and equation (18) becomes

$$y = -x\frac{\cos(\omega)}{\sin(\omega)} + \frac{a\cos(\omega - \theta) - \rho}{\sin(\omega)} \ .$$

Thus, the equation of the line normal to the line tangent to the dog's path is

$$x\cos(\omega) + y\sin(\omega) = a\cos(\omega - \theta) - \rho \ . \qquad (19)$$

We can complete our analysis by first differentiating equations (17) and (19) with respect to θ. For (17) this gives

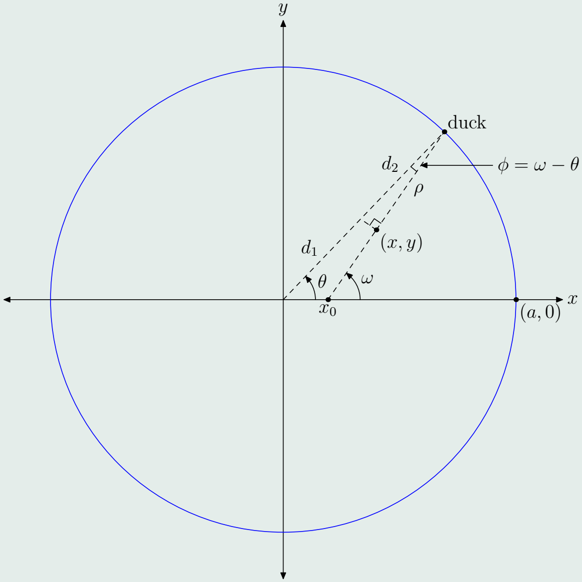


Figure 7: The geometry of the normal line.

$$\frac{dx}{d\theta}\sin(\omega) + x\cos(\omega)\frac{d\omega}{d\theta} - \frac{dy}{d\theta}\cos(\omega) + y\sin(\omega)\frac{d\omega}{d\theta} = a\cos(\omega-\theta)\left(\frac{d\omega}{d\theta} - 1\right) ,$$

or, grouping the $d\omega/d\theta$ terms together and factoring,

$$\frac{dx}{d\theta}\sin(\omega) - \frac{dy}{d\theta}\cos(\omega) + \frac{d\omega}{d\theta}[x\cos(\omega) + y\sin(\omega)] = a\cos(\omega-\theta)\left(\frac{d\omega}{d\theta} - 1\right) .$$

By equation (19), the factor in square brackets on the left-hand side is $a\cos(\omega-\theta) - \rho$, so we have

$$\frac{dx}{d\theta}\sin(\omega) - \frac{dy}{d\theta}\cos(\omega) + a\cos(\omega-\theta)\frac{d\omega}{d\theta} - \rho\frac{d\omega}{d\theta} = a\cos(\omega-\theta)\frac{d\omega}{d\theta} - a\cos(\omega-\theta)$$

or,

$$\frac{dx}{d\theta}\sin(\omega) - \frac{dy}{d\theta}\cos(\omega) - \rho\frac{d\omega}{d\theta} = -a\cos(\omega-\theta) . \tag{20}$$

Now, the variables x and y, at the dog's position, must satisfy the pursuit curve, the tangent line to the curve, and the normal to the tangent line, and we can see from figure 8 that x and y are related by

$$\frac{dy}{dx} = \tan(\omega) .$$

We know that, at any given time, the duck has moved through a distance of $a\theta$ (refer back to figure 6), and since the dog travels at a speed n times that of the duck, the distance traveled by the dog in the same amount of time will be $s = an\theta$, which tells us that $ds = and\theta$. From figure 8 we see that

$$\frac{dx}{ds} = \cos(\omega)$$

$$ds = \frac{dx}{\cos(\omega)} ,$$

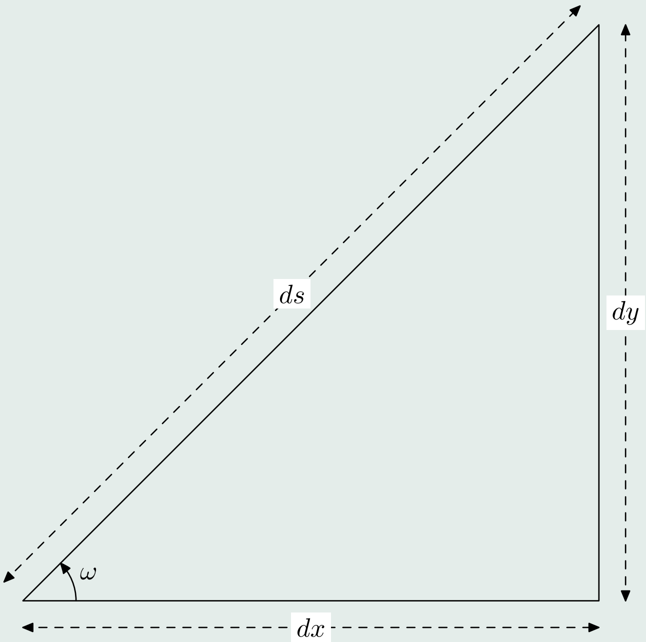


Figure 8: The differential triangle relating ds, dx, and dy.

so

$$\frac{dx}{\cos(\omega)} = and\theta$$

and

$$\frac{dx}{d\theta} = an\cos(\omega)\ . \tag{21}$$

We know, from the chain rule in calculus,

$$\frac{dy}{d\theta} = \frac{dy}{dx} \cdot \frac{dx}{d\theta} = tan(\omega)an\cos(\omega)\ ,$$

so we have

$$\frac{dy}{d\theta} = an\sin(\omega)\ . \tag{22}$$

Inserting (21) and (22) into (20), we see

$$an\cos(\omega)\sin(\omega) - an\sin(\omega)\cos(\omega) - \rho\frac{d\omega}{d\theta} = -a\cos(\omega - \theta)\ ,$$

and now we finally arrive at the differential equation

$$\boxed{\rho\frac{d\omega}{d\theta} = a\cos(\omega - \theta)}\ . \tag{23}$$

We have two variables which describe the dog's position, ω and ρ, so we get our second differential equation by differentiating equation (19), which gives us

$$\frac{dx}{d\theta}\cos(\omega) - x\sin(\omega)\frac{d\omega}{d\theta} + \frac{dy}{d\theta}\sin(\omega) + y\cos(\omega)\frac{d\omega}{d\theta} = -a\sin(\omega - \theta)\left(\frac{dw}{d\theta} - 1\right) - \frac{d\rho}{d\theta}\ ,$$

or

$$\frac{dx}{d\theta}\cos(\omega) + \frac{dy}{d\theta}\sin(\omega) - \frac{d\omega}{d\theta}[x\sin(\omega) - y\cos(\omega)] = -a\sin(\omega-\theta)\left(\frac{d\omega}{d\theta} - 1\right) - \frac{d\rho}{d\theta} \ .$$

By equation (17), the factor in square brackets is $a\sin(\omega - \theta)$, so

$$\frac{dx}{d\theta}\cos(\omega) + \frac{dy}{d\theta}\sin(\omega) = a\sin(\omega-\theta) - \frac{d\rho}{d\theta} \ .$$

Inserting (21) and (22) as before gives us

$$an\cos^2(\omega) + an\sin^2(\omega) = a\sin(\omega-\theta) - \frac{d\rho}{d\theta} \ ,$$

which leads to

$$an(\cos^2(\omega) + \sin^2(\omega)) = asin(\omega-\theta) - \frac{d\rho}{d\theta} \ ,$$

which leads us to our second differential equation,

$$\boxed{\frac{d\rho}{d\theta} = a[\sin(\omega-\theta) - n]} \ . \tag{24}$$

Referring back to figure 7, we can write $\phi = \omega - \theta$, solving for ω and differentiating with respect to θ, we get

$$\omega = \phi + \theta$$
$$\frac{d\omega}{d\theta} = \frac{d\phi}{d\theta} + 1 \ .$$

By equation (23), we can write

$$\rho\left(\frac{d\phi}{d\theta} + 1\right) = a\cos(\phi)$$

$$\rho \frac{d\phi}{d\theta} + \rho = a\cos(\phi) \ . \tag{25}$$

Differentiating equation (24) with respect to θ gives us

$$\frac{d^2\rho}{d\theta^2} = a\cos(\phi)\frac{d\phi}{d\theta} \ ,$$

which, when solved for $d\phi/d\theta$, becomes

$$\frac{d\phi}{d\theta} = \frac{d^2\rho/d\theta^2}{\cos(\phi)} \ .$$

Substituting this in equation (25) gives us

$$\rho \frac{d^2\rho}{d\theta^2} + a\rho\cos(\phi) = a^2\cos^2(\phi) \ . \tag{26}$$

Now, being on a circular limit cycle means that, once the dog reaches its limit cycle, the distance ρ between the dog and the duck will remain constant. This means that, as $\theta \to \infty$, $\lim \rho = c$. This also tells us that

$$d\rho/d\theta = d^2\rho/d\theta^2 = 0 \ . \tag{27}$$

Substituting this in equation (25), we find that as $\theta \to \infty$,

$$\begin{aligned} \rho &= a\cos(\phi) \\ \cos(\phi) &= \frac{\rho}{a} \ , \end{aligned}$$

and, from equation (**??**), as $\theta \to \infty$, $\sin\phi = n$. Inserting these results into equation (26), we find that, once the dog has reached its limit cycle,

$$\begin{aligned} a\rho\left(\frac{\rho}{a}\right) &= a^2(1 - \sin^2(\phi)) \\ \rho^2 &= a^2(1 - n^2) \ . \end{aligned}$$

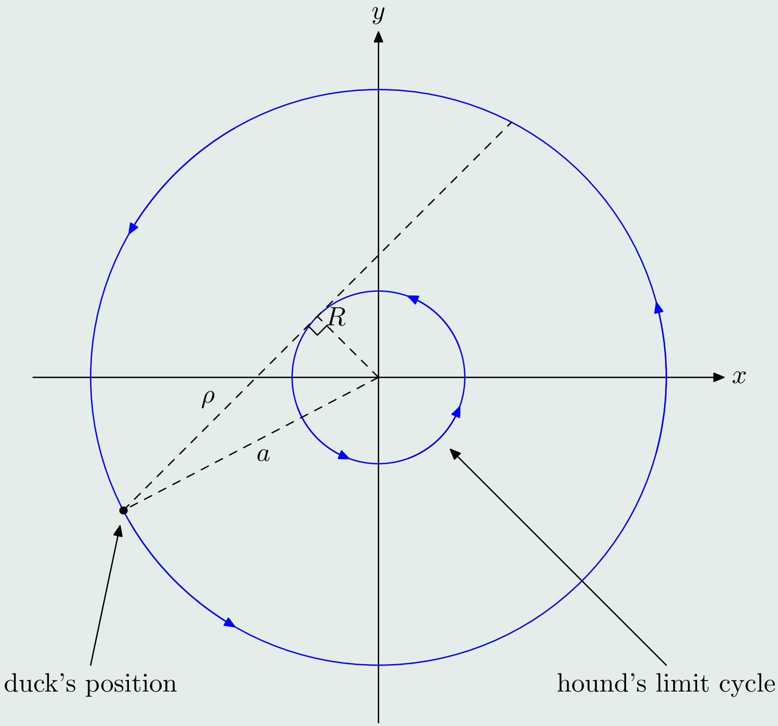


Figure 9: Geometry of the limit cycle.

So, we can conclude that the constant distance between the duck on its circular path and the dog on its circular limit cycle is given by

$$\rho = a\sqrt{1-n^2}\ .$$

Further, if we let R be the radius of the limit cycle as in figure 9, using the Pythagorean theorem shows that $R^2 + \rho^2 = a^2$, so

$$\begin{aligned} R &= \sqrt{a^2-\rho^2} \\ R &= \sqrt{a^2 - a^2(1-n^2)} \\ R &= an\ . \end{aligned}$$

This tells us that, if $n < 1$, the curve of pursuit of the dog eventually settles down to a circular limit cycle, the radius of which is n times the radius a of the duck's circular path.

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