Fish Population Modeling

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Abstract

Fish are essential to the economy because they are a good source of food and many people have jobs as fishermen. It is possible to model the behavior of a given fish population under certain conditions. There are many factors that can be considered: the conditions of their habitat, such as water quality and temperature, the starting population, how frequently they are harvested, and the probability that the fish are eaten by a predator. Using a model, we can figure out when and how many of the fish can be harvested to maximize the amount of food obtained without completely depleting the population. This report will explore the logistic growth model with a fish population and how fish are harvested safely.

1. Logistic Model of a Fishing Season

A fish population can be modelled using the Logistic Equation, \( P' = kP(1 - P/a) \), where \( P \) is the population at time \( t \), \( k \) is a growth constant, and \( a \) is the limiting value, or carrying capacity. However, a population of fish can be either harvested or stocked. Therefore the differential equation becomes

\[ P' = kP(1 - P/a) + s - h, \]  

(1)
where \( s \) is the amount of fish stocked and \( h \) is the amount of fish harvested\(^2\). The parameters \( s \) and \( h \) can either be constant values or functions of time. If no fish are stocked or harvested, the population has the exact form of the logistic equation and has equilibrium points at \( a \), the limiting value, and at zero. This is shown in Figure 1. Any initial population of fish will eventually reach the equilibrium point of 15 tons. Obviously, stocking the population will cause it to grow and will raise the equilibrium value. Harvesting fish, however, is detrimental to the population. If too many fish are harvested, the population can become extinct.

We can use qualitative analysis to figure out how many fish can be harvested and still allow the population to survive. We will concentrate on the value of \( h \) and how it affects the equilibrium points. As \( h \) gets larger, or as more fish are being harvested, the equilibrium points move closer together, as shown in Figure 2. We can solve for these equilibrium points by setting the right-hand side of (1) equal to zero and solving for \( P \). When fish are being harvested, if the initial population is too small, the population will become extinct. If \( h \) were to keep increasing, the equilibrium points would continue to get closer together until eventually they would meet and there would only be one equilibrium point that is semi-stable. This is called bifurcation and is shown in Figure 3. Any initial population above the equilibrium point will decrease to the equilibrium value and any initial population below the equilibrium point will decrease to zero. If the harvesting is increased past the point of bifurcation, the fish population will die off no matter what.

2. **Limited Harvesting**

With uncontrolled fish harvesting, a population could easily become extinct. That’s why it is common to limit the amount of fish that can be harvested or to only allow fishing at certain times during the year. A time in which there is no harvesting lets
Figure 1: Fish Population, $h = 0$
Figure 2: Fish Population, $h = 1$
Figure 3: Bifurcation

\[ P' = kP(1 - \frac{P}{a}) - h \]

\[ k = 0.5 \quad a = 15 \quad h = 1.875 \]
the population recover and reproduce so that it can remain above the critical value. For example, the government could limit fishing to only the first six months of the year and make it illegal to fish during the second half of the year. Therefore, \( h \) would be a piecewise function, and would be equal to 2 tons of fish per month during January through June, and zero during July through December, or

\[
\begin{align*}
2, & \quad \text{if } t \leq 6 \\
0, & \quad \text{if } t > 6.
\end{align*}
\]

As seen in Figure 4, the population decreases for the first six months, then increases until it reaches the limiting value of 15 tons. However, the population could become extinct if the initial population is too small. Therefore, when making fishing regulations, it is important to limit harvesting so that the population can survive and be able to have a sufficient period of growth.

It is also common to have some months where heavy fishing is allowed and other months where only light fishing is allowed. An example of this is allowing heavy fishing for the first three months of the year and less fishing during the rest of the year, or \( h \) is equal to

\[
\begin{align*}
4, & \quad \text{if } t \leq 3 \\
1, & \quad \text{if } t > 3.
\end{align*}
\]

In Figure 5 the population still recovers to the equilibrium value. However, it takes longer for it to reach the stable equilibrium because there are still a few fish that are being harvested during the rest of the year.

### 3. Fishing Licenses

Another way the government can limit harvesting is by issuing licenses. By only allowing a certain number of licenses and limiting the amount of fish harvested by each license
Figure 4: Fish Population, $h = 2(t \leq 6)$
Figure 5: Fish Population, $h = 4(t \leq 3) + 1(t > 3)$
holder, they will know how many fish are legally harvested in a given amount of time. The harvest term of the equation then becomes \( h = nL \), where \( L \) is the number of licenses issued and \( n \) is the number of fish each license-holder is allowed to catch. For example, the equation

\[
P' = 0.25P(1 - \frac{P}{40000}) - 20L
\]

models a fish population with a growth constant of 0.25, a carrying capacity of 40,000 fish, and allows \( L \) licence-holders to catch 20 fish each. If we graph \( P \) versus \( P' \) for several values of \( L \), we can figure out where bifurcation occurs and limit the number of licenses accordingly. In Figure 2, the graph of \( P \) gets lower and lower as \( L \) increases and the \( P \)-intercepts are the equilibrium values. When \( L = 125 \), there is only one \( P \)-intercept and only one equilibrium point, which means this is the bifurcation point. Any \( L \) value greater than 125 will result in eventual extinction of the population. Therefore, for this example, less than 125 licenses should be issued. To be safe, only about 100 licenses should be issued to account for uncontrollable things like illegal fishing, poor reproduction, and disease to make sure that the population survives.

4. Periodic Harvesting

The previous example with licenses can be expanded to account for periodic harvesting. The differential equation would then look something like:

\[
P' = 0.25P(1 - \frac{P}{40000}) - 20L(1 + \sin 2\pi t)
\]

The question now is if this periodic equation has the same bifurcation value of \( L = 125 \). We can’t use the same approach of plotting \( P \) versus \( P' \) that we used in the previous section to find the equilibrium points for various values of \( L \) because equation (3) is non-autonomous, meaning that the right-hand side is dependent on the time \( t \) [4]. As
Figure 6: Graphs of several values of $L$ for equation (2)
Figure 7: Solutions for $L = 50$

Mathematical expression:

\[ P' = k P \left( 10^4 P / a \right) - 20 L \left( 1 + \sin(2\pi t) \right) \]

\[ k = 0.25 \quad a = 40000 \quad L = 50 \]
seen in Figure 7, the solutions are periodic and they have the same general trend as the solutions in the previous section. There are two solutions that oscillate about the equilibrium values. The solutions converge to one periodic solution that oscillates around the stable equilibrium point. We can see this by zooming in on the periodic solutions at a high value of $t$. In Figure 8, we see that all of the solutions have a period of exactly 1. This is what we would expect, because $T = 2\pi/2\pi = 1$. In Figure 9, the number of licenses is increased to 100. Just like before, the equilibrium points get closer together. In Figure 10 the equilibrium points are closer still. Figure 11 shows that for a value of $L = 125$, there is only one equilibrium point. It appears that this periodic equation has the same bifurcation value as the autonomous equation. In Figure 12, we show a solution with $L = 125$ and a starting population at the carrying capacity of 40,000 fish at $t = 0$ that continues for 500 years. The population reaches the equilibrium point and stays there. However, if we allow just one more license and increase $L$ to 126, we will be past bifurcation and the population will die off no matter what. This is shown in Figure 13. Therefore, this periodic equation has the same bifurcation value as equation (2).

5. Analysis of a Non-Autonomous Differential Equation

In the previous section, the differential equation $P' = 0.25P(1-\frac{P}{40000})-20L(1+sin2\pi t)$ had the non-autonomous term $sin2\pi t$. It can be shown that the non-autonomous term $x(t) = sin2\pi t$ is a part of the solution to the system

$$\begin{cases}
\frac{dx}{dt} = 2\pi y, & x(0) = 0 \\
\frac{dy}{dt} = -2\pi x, & y(0) = 1,
\end{cases}$$
Figure 8: Solutions converge to one solution with $T=1$
Figure 9: Solutions for L=100
Figure 10: Solutions for L=120
Figure 11: Solutions for L=125

\[ P' = k P \left( \frac{1}{10^4} \frac{P}{a} \right) - 20 L \left( 1 + \sin(2 \pi t) \right) \]

\[ k = 0.25 \quad a = 40000 \quad L = 125 \]
Figure 12: Solution for L=125

\[ P' = kP \left(\frac{10^4 P}{a}\right) - 20L \left(1 + \sin(2\pi t)\right) \]

\[ k = 0.25, \quad a = 40000, \quad L = 125 \]
Figure 13: Solution for $L=126$
which can also be written as
\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix} 0 & 2\pi \\ -2\pi & 0 \end{bmatrix} \begin{bmatrix} x \\
y
\end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\
1
\end{bmatrix}.
\]

This system has eigenvalues $\pm 2\pi i$ and general solution
\[
\mathbf{x} = C_1 \begin{bmatrix} \cos 2\pi t \\
-\sin 2\pi t
\end{bmatrix} + C_2 \begin{bmatrix} \sin 2\pi t \\
\cos 2\pi t
\end{bmatrix}.
\]

Our initial condition gives
\[
\begin{bmatrix} 0 \\
1
\end{bmatrix} = C_1 \begin{bmatrix} 1 \\
0
\end{bmatrix} + C_2 \begin{bmatrix} 0 \\
1
\end{bmatrix}.
\]

Therefore, $C_1 = 0$ and $C_2 = 1$ and the solution to the initial condition is
\[
\mathbf{x} = \begin{bmatrix} \sin 2\pi t \\
\cos 2\pi t
\end{bmatrix},
\]
or $x = \sin 2\pi t$ and $y = \cos 2\pi t$.

Now we can add a third equation to our system to get the following three-dimensional system that is autonomous.
\[
\begin{cases}
x' = 2\pi y, & x(0) = 0 \\
y' = -2\pi x, & y(0) = 1 \\
P' = 0.25P(1 - P/40000) - 20L(1 + x), & P(0) = 40000
\end{cases}
\tag{4}
\]

Setting $x$ and $y$ both equal to zero, we can now set the equation $P' = 0.25P - P^2/160000 - 20L$ equal to zero and solve for the equilibrium points in the same way that we could for equation (2). For example, if we substitute 75 for $L$, we get $P$ values
Figure 14: Solution to the autonomous system (4) for L=75
Figure 15: Solution to system (4) in three dimensions
of 32,650 and 7,350. In Figure 14, $P$ starts at 40,000 at $t = 0$ and declines to the equilibrium value of 32,650 fish while $x$ and $y$ oscillate between positive and negative 1. This is shown in three dimensions in Figure 15. In Figure 15, the distance from the $P$ axis to the curve is $C(x, y) = x^2 + y^2$. To find the rate of change of $C$, we differentiate with respect to $t$.

$$\frac{dC}{dt} = 2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right)$$

$$= 2x(2\pi y) + 2y(-2\pi x)$$

$$= 4\pi xy - 4\pi xy$$

$$= 0$$

Since $C$ is not changing as time goes by, the distance between the curve and the $P$ axis doesn’t change and the solutions are in the shape of cylinders and either increase or decrease.

6. Fish Population with Predators

In a larger body of water, there might be a predator that kills off some of a specific fish population. In general, the population grows depending on the growth constant. However, with a predator, the fish population will also decrease by a certain amount, depending on the population of the predator. Therefore, the population of fish is given by the equation $F' = aF - bFP$, where $F$ is the fish population, $P$ is the predator population, and $a$ and $b$ are constants. Assuming fish is its only prey, the population of the predator will decrease if there are no fish and will increase proportional to the number of fish that are present. Therefore, the population of the predator is given by the equation $P' = -cP + dFP$, where $F$ is the fish population, $P$ is the predator
population, and $c$ and $d$ are constants.

Now that we have two equations, we can plot $F'$ vs. $P'$ in the phase plane. This is shown in Figure 16. The initial fish population is 100 and the initial predator population is 40. Following the arrows, the fish population decreases and the predator population increases because the predator eats some of the fish. The predator population reaches a maximum of just less than 80. Then both populations decrease because there are less fish for the predators to eat so some of them starve and die. Eventually, the predator population becomes small enough that the fish can start to reproduce and the population starts to grow. The process continues until we get back to the point (100,40) and it starts over again.

We can also use the predator-prey model with harvesting. The Volterra Model, which was originally based on harvested fish populations in the Adriatic Sea [3], is given by the system:

\[
\begin{align*}
F' &= aF(1 - bP) - H_1 F \\
P' &= cP(-1 + dF) - H_2 P,
\end{align*}
\]

where $H_1$ and $H_2$ are the respective harvest functions of the fish and the predator, assuming that both the fish and predator are being harvested.

7. Growth of an Individual Fish

There are several models that have been developed for the growth of an individual fish in a particular population. One of these is the von Bertalanffy model[5], which is given by the differential equation

\[
\frac{dW}{dt} = \phi W^{2/3} - pW,
\]
Figure 16: Phase Plot of Predator vs. Prey
where \( W \) is the weight of the fish, \( \phi \) is the fish’s rate of feeding, and \( p \) is the fish’s metabolic rate. Although this equation is difficult to solve, \( W \) can be plotted as a function of time using one of Matlab’s differential equation solvers, such as ode45. Figure 17 shows that the fish begins at a weight of half a pound, grows rapidly at first, then the growth plateaus at just over four and a half pounds. In Figure 18, the value of \( \phi \) is slightly greater while the metabolic rate remains the same, which means the fish has more food to eat. Therefore, the fish will grow to be larger, about six pounds.

The length of a fish can also be modeled using the equation

\[
\frac{dL}{dt} = \frac{p}{3}(L_{\text{max}} - L),
\]

where \( L \) is the length of the fish and the maximum length, \( L_{\text{max}} \), is equal to \( \phi/(p\lambda^{1/3}) \). This assumes isometric growth, or that \( W = \lambda L^3 \).

8. Water Quality

In order for a fish population to survive, it is important that the body of water the fish are living in is a suitable environment. One important aspect of water quality is the oxygen concentration. The differential equation \( \frac{dO}{dt} = K(O_{eq} - O) \) models the oxygen content of a body of water. The concentration of oxygen is given by \( O \), \( O_{eq} \) is the equilibrium concentration, and \( K \) is the reaeration coefficient, which is a measure of how well oxygen moves from the air to the water\(^2\). In Figure 19, the oxygen concentration reaches equilibrium and stays there.

The equation can be expanded to \( \frac{dO}{dt} = K(O_{eq} - O) \pm S \), where \( S \) represents either an oxygen source or something that takes away oxygen. For example, an oxygen source would be plants (photosynthesis) and any human or animal would take away oxygen by breathing it in. Figure 20 shows what occurs with the extra parameter \( S \). Since \( S \) is positive in this example, there is an oxygen source, which makes the oxygen
Figure 17: von Bertalanffy Model of Fish Growth

\[ \phi = 0.5, \ p = 0.3 \]
Figure 18: von Bertalanffy Model of Fish Growth
Figure 19: Oxygen Concentration in Water

$O_{eq} = 5, S=0, K=1$
Figure 20: Oxygen Concentration with extra Oxygen Source

$O_{eq} = 7, S=0.4, K=0.2$
concentration higher than the equilibrium concentration. Also, notice that the $K$ value is less than in Figure 19. This makes the water take longer to reach its equilibrium oxygen concentration.

References


