Resistance is Futile
Factoring Air Resistance into Projectile Motion

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Introduction

The goal of this presentation is to explain how to set up, solve, and compare two commonly used models of projectile motion through two dimensions: a model which assumes negligible air resistance, and a model which assumes linear air resistance.
Linear Resistance

\[ F_{\text{drag}} = bV \]

As the term implies, the linear model fairly assumes that the force of drag is directly proportional to projectile velocity - a model that usually works well at low velocity, where viscous drag dominates.
Negligible Air Resistance

\[ F_{\text{drag}} = 0 \]

This model is usually presented in introductory physics classes and derived with kinematics equations.
When describing projectile motion through two dimensions, two components of motion must be considered: the horizontal component of motion and the vertical component.
1) Analyze the horizontal and vertical components of motion separately.
Strategy for Deriving Trajectory

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2) Derive an equation for horizontal displacement as a function of time.
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2) Derive an equation for horizontal displacement as a function of time.
3) Derive an equation for vertical displacement as a function of time.
4) Solve the horizontal displacement for time.
Strategy for Deriving Trajectory

1) Analyze the horizontal and vertical components of motion separately.
2) Derive an equation for horizontal displacement as a function of time.
3) Derive an equation for vertical displacement as a function of time.
4) Solve the horizontal displacement equation for time.
5) Substitute that expression of time into the vertical displacement equation, resulting in vertical displacement as a function of horizontal displacement (trajectory).
Resistance is Futile

Negligible Air Resistance: Horizontal Motion

With no air resistance, there is no force acting in the horizontal direction, so the projectile’s horizontal velocity remains constant for as long as its in flight. From kinematics, horizontal displacement:

\[ x = V_{x0} t \]

\[ = V_0 \cos(\theta) t \]

Initial velocity vector and its components.
In the vertical direction, the acceleration of gravity will cause the projectile’s vertical velocity to decrease throughout its flight. From kinematics, vertical displacement:

\[ y = V_{y0} t - \frac{1}{2} gt^2 \]

\[ = V_0 \sin(\theta) t - \frac{1}{2} gt^2 \]

**Initial velocity vector and its components.**
Negligible Air Resistance: Trajectory

Horizontal displacement: \[ x = V_{x_0} t \]

Vertical displacement: \[ y = V_{y_0} t - \frac{1}{2} gt^2 \]
Negligible Air Resistance: Trajectory

Horizontal displacement:

\[ x = V_{x_0} t \]

Solve for time (t):

\[ t = \frac{x}{V_{x_0}} \]

Vertical displacement:

\[ y = V_{y_0} t - \frac{1}{2} gt^2 \]
Horizontal displacement:

\[ x = V_{x_0} t \]

Solve for time (t):

\[ t = \frac{x}{V_{x_0}} \]

Vertical displacement:

\[ y = V_{y_0} t - \frac{1}{2} gt^2 \]

Substitute into vertical displacement:

\[ Y_{vac} = V_{y_0} \left( \frac{x}{V_{x_0}} \right) - \frac{g}{2} \left( \frac{x}{V_{x_0}} \right)^2 \]
Negligible Air Resistance: Trajectory

Horizontal displacement:

\[ x = V_{x_0} t \]

Solve for time (t):

\[ t = \frac{x}{V_{x_0}} \]

Simplify to reveal:

\[ Y_{vac} = x \tan \theta - \frac{gx^2}{2V_0^2 \cos^2 \theta} \]

Vertical displacement:

\[ y = V_{y_0} t - \frac{1}{2} gt^2 \]

Substitute into vertical displacement:

\[ Y_{vac} = V_{y_0} \left( \frac{x}{V_{x_0}} \right) - \frac{g}{2} \left( \frac{x}{V_{x_0}} \right)^2 \]
Trajectory With Negligible Air Resistance

\[ V_0 = 60 \text{m/s}, \theta = 45^\circ \]
The same strategy will be used to derive trajectory through air:
1) Analyze the horizontal and vertical components of motion separately.
2) Derive an equation for horizontal displacement as a function of time.
3) Derive an equation for vertical displacement as a function of time.
4) Solve the horizontal displacement equation for time.
5) Substitute that expression of time into the vertical displacement equation, resulting in vertical displacement as a function of horizontal displacement (trajectory).
Linear Air Resistance: Horizontal Net Force

Begin with an analysis of the forces acting in the horizontal direction, where the only force is the force of air resistance. Begin with the Newtonian equation:

\[ \sum F_x = ma_x \]

\[ ma_x = -bV_x \]

\[ m\frac{dV_x}{dt} = -bV_x \]

\[ \frac{dV_x}{dt} = -\frac{bV_x}{m} \quad (1) \]

Equation (1) is the differential equation to be solved for \( V_x \).
At $t = 0$, the velocity in the horizontal direction, $V_{x0}$, will be equal to the initial velocity times the cosine of the initial projection angle ($V_0 \cos \theta$). These initial conditions complete the setup of the differential equation for horizontal velocity.

$$\frac{dV_x}{dt} = -\frac{bV_x}{m}, \quad V_x(0) = V_{x0}$$
Solving by separation of variables shows:

\[ V_x(t) = V_x e^{-bt/m} = V_0 \cos(\theta) e^{-bt/m} \]

Horizontal velocity, where
\[ V_0 = 60 \text{m/s}, \quad \theta = 45^\circ, \quad m = 1 \text{kg}, \quad b = 1 \text{kg/s}. \]
Linear Air Resistance: Horizontal Displacement

Since velocity is the time derivative of displacement, integrating the equation for horizontal velocity yields horizontal position:

\[
x(t) = \int_0^t V_{x0} e^{-bt'/m} \, dt'
= \frac{mV_{x0}}{b} \left(1 - e^{-bt/m}\right)
\]

Horizontal position, where \(V_0 = 60\text{m/s}, \theta = 45^\circ, m = 1\text{kg}, b = 1\text{kg/s} \).
Linear Air Resistance: Vertical Net Force

To derive the differential equation of vertical velocity, refer again to the free body diagram and set up the Newtonian equation:

\[ \sum F_y = ma_y \]

\[ ma_y = -mg - bV_y \]

\[ m \frac{dV_y}{dt} = -mg - bV_y \]

\[ \frac{dV_y}{dt} = -g - \frac{bV_y}{m} \]
When $t = 0$, the projectile is moving in the vertical direction with a velocity equal to the initial velocity times the sine of the projected angle ($V_0 \sin \theta$). For ease, $V_{y0}$ will be used to express this.

$$\frac{dV_y}{dt} = -g - \frac{b V_y}{m}, \quad V_y(0) = V_{y0}$$
Solving, again by separation of variables, shows:

$$V_y(t) = \left( \frac{mg}{b} + V_{y0} \right) e^{-bt/m} - \frac{mg}{b}$$

Vertical velocity, where

$$V_0 = 60\text{m/s}, \theta = 45^\circ, m = 1\text{kg}, b = 1\text{kg/s}.$$
Linear Air Resistance: Vertical Displacement

Attain an equation for vertical displacement by integrating the equation of vertical velocity:

\[ y(t) = \int_0^t \left( \left( \frac{mg}{b} + V_{y0} \right) e^{-bt'/m} - \frac{mg}{b} \right) dt' \]

\[ = \left( \frac{m^2g + bmV_{y0}}{b^2} \right) \left( 1 - e^{-bt/m} \right) - \frac{mg}{b} t \]

Vertical position, where
\[ V_0 = 60 m/s, \ \theta = 45^\circ, \ m = 1 kg, \ b = 1 kg/s. \]
Linear Air Resistance: Trajectory

Horizontal Displacement: \[ x = \frac{mV_{x_0}}{b}(1 - e^{-bt/m}), \]

Vertical Displacement: \[ y = \left( \frac{m^2g + bmV_{y_0}}{b^2} \right) (1 - e^{-bt/m}) - \frac{mg}{b} t \]
Linear Air Resistance: Trajectory

\[ x = \frac{mV_{x0}}{b}(1 - e^{-bt/m}), \quad y = \left(\frac{m^2g + bmV_{y0}}{b^2}\right)(1 - e^{-bt/m}) - \frac{mg}{b} t \]

Solve horizontal displacement for time \((t)\):

\[ t = \frac{m}{b} \ln \left(\frac{mV_{x0}}{mV_{x0} - bx}\right) \]
Linear Air Resistance: Trajectory

\[ x = \frac{mV_{x0}}{b} (1 - e^{-bt/m}), \quad y = \left( \frac{m^2g + bmV_{y0}}{b^2} \right) (1 - e^{-bt/m}) - \frac{mg}{b} t \]

Solve horizontal position for time (\( t \)):

\[ t = \frac{m}{b} \ln \left( \frac{mV_{x0}}{mV_{x0} - bx} \right) \]

Substitute this expression of \( t \) into the equation for vertical position to produce an equation for trajectory (\( Y \)).

\[ Y = \left( \frac{mg}{bV_{x0}} + \frac{V_{y0}}{V_{x0}} \right) x + \frac{m^2g}{b^2} \ln \left( 1 - \frac{b}{mV_{x0}} x \right) \]
Linear Air Resistance: Trajectory

Expand $V_{x_0}$, and $V_{y_0}$:

$$Y = \left( \frac{mg \sec \theta}{bV_0} + \tan \theta \right) x + \frac{m^2 g}{b^2} \ln \left( 1 - \frac{b \sec \theta}{mV_0} x \right)$$
What is $b$?

Recall the differential equation for the vertical component of velocity:

$$\frac{dV_y}{dt} = -g - \frac{bV_y}{m}$$
An object dropped from rest and allowed to fall freely will accelerate until reaching its terminal velocity, at which point acceleration equals zero. Under these conditions, and letting the y-axis point downward, we can write:

\[
\frac{dV_y}{dt} = -g - \frac{bV_y}{m}
\]

\[
0 = g - \frac{bV_{term}}{m}
\]
What is $b$?

Solving for $b$ shows:

$$b = \frac{mg}{V_{term}}$$
What is $b$?

Now, consider a 3.2 gram paintball with an experimentally derived terminal velocity of 21.8 meters per second:

$$b = \frac{mg}{V_{\text{term}}} = \frac{(0.0032\, \text{kg})(9.81\, \text{m/s}^2)}{21.8\, \text{m/s}} = 0.00144\, \text{kg/s}$$
Recall the equation for trajectory:

\[ Y = \left( \frac{mg \sec \theta}{bV_0} + \tan \theta \right) x + \frac{m^2 g}{b^2} \ln \left( 1 - \frac{b \sec \theta}{mV_0} x \right) \]
Trajectory of a Paintball

With values:
\[ V_0 = 90 \text{ m/s}, \]
\[ \theta = 45^\circ, \]
\[ m = 0.0032 \text{ kg}, \]
\[ b = 0.00144 \text{ kg/s}, \]
\[ g = 9.81 \text{ m/s}^2 \]

The equation for a paintball’s trajectory becomes:

\[ Y = 1.343x + 48.44 \ln(1 - 0.007071x) \]
Trajectory of a 3.2 gram paintball fired at 90 m/s at 45 degrees.
Trajectory of a Paintball With and Without Air Resistance
Maximum Range Angles

A Paintball's Trajectory with & without Air Resistance

- Without Air, 45 degrees
- With Air, 45 degrees
- Without Air, 24 degrees
- With Air, 24 degrees
Increase Mass: 3.2g
Increase Mass: 10g
Increase Mass: 15g
Increase Mass: 25g
Increase Mass: 50g
Increase Mass: 100g
Increase Mass: 10kg
But a higher mass doesn’t imply it’s OK to assume air resistance is negligible, because $b$ changes with mass. Consider a 7.62kg bowling ball with a terminal velocity of 83.1m/s:

$$b = \frac{mg}{V_{\text{term}}}$$

$$= \frac{(7.62\text{kg})(9.81\text{m/s}^2)}{83.1\text{m/s}}$$

$$= 0.8995\text{kg/s}$$
Bowling Ball Trajectory

A Bowling Ball's Trajectory with & without Air Resistance

- Without Air
- With Air, $m=7.62$ kg, $b=0.8995$
(1) Pruneau, Claude A. "Projectile Motion." Wayne State University.
http://rhig.physics.wayne.edu/pruneau/CoursesPHY5200/lectures/PHY5200-Chap2.pdf