Modeling the Spruce Budworm Outbreak Using Differential Equations

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Brief Description

The Spruce Budworm is one of the most destructive native insect in the northern spruce and fir forests of the Eastern United States and Canada. Majority of the time, the number of budworms remains at a constant low level. However, every few decades, the population of budworms increases to a huge population, depleting the forest and destroying many trees, before dropping back down to its normal population level. Evidence suggests these outbreaks have been repeating itself regularly for hundreds, if not thousands, of years.
Forest Depletion Due To Budworm
Why Is This Important

Fir is highly vulnerable to insect attacks and rot. For this reason, it is generally used for indoor applications such as paneling and light frame construction. Red, white and black spruce are cut into lumber, utility poles, pilings, boat building stock, furniture, boxes and crates. These species are also used to make plywood and flake board. Sitka spruce is used to make sounding boards for high-quality pianos, guitar faces, ladders and components of experimental light aircraft. Also the most common use for this tree are christmas trees.
Logistic Model

The model for budworm population size can be model by the *logistic model equation*. Let $t$ be time and let $N$ be the budworm population size. We model the evolution of the Budworm according to the differential equation of the form

$$\frac{dN}{dt} = r \left( 1 - \frac{N}{K} \right) N,$$

where $r$ is the growth rate, $N$ is the population size, and $K$ is the carrying capacity.
Predation

The rate of predation is highly dependent on the budworm’s population. The predator model is modeled by the equation

\[ p(N) = \frac{BN^2}{A^2 + N^2}, \]

where \( A \) and \( B \) are constants.
Several Important Points

There are a few important points you need to know about the model.

1. If the spruce budworm population is small, the predation rate is low, if not close to zero, since the predators will pursue other prey.

2. The predation rate will grow as the budworms become more numerous.

3. Predation rate cannot grow to become too large.
Proof

If we take the limit of our equation as \( N \) approaches infinite

\[
\lim_{{N \to \infty}} \frac{BN^2}{A^2 + N^2} = \lim_{{N \to \infty}} \frac{B}{\frac{A^2}{N^2} + 1} = B.
\]

Here we see that \( B \) represents the boundary of the death rate of the Budworm due to predation.
Rate of increase

If we take the derivative of \( p(N) \) we will be able to calculate the rate at which the predation increase. I calculated my answer to be

\[
p'(N) = \frac{2A^2BN}{(A^2 + N^2)^2}.
\]
By deriving for a second time and setting $p''(N) = 0$, we can solve for $N$ and get our critical point. $N$ will give us our critical number which represents where our graph shifts concavity from up to down. I calculated

$$p''(N) = \frac{2A^2 B [A^2 - 3N^2]}{(A^2 + N^2)^3}.$$

and

$$p''(0) = \frac{A}{\sqrt{3}}.$$
Predator Curve

\[ p(N) \]

\[ A = \frac{1}{\sqrt{3}} \]

\[ B \]
Equilibrium Points

We can rewrite our original equation \( r(1 - \frac{u}{q}) - \frac{u^2}{1+u^2} \) by subtracting \( p(n) \) from both sides.

\[
r(1 - \frac{u}{q}) = \frac{u^2}{1 + u^2}
\]

If we combine the graphs of both equations we get the graph.

\[
r(1 - u/q) = u/(1 + u^2)
\]
By combining both equations and taking $u'$ we get

$$u' = ru \left(1 - \frac{u}{q}\right) - \frac{u^2}{1 + u^2}$$
Here we have our vector plot of our equation

\[ u' = ru(1 - u/q) - u^2/(1 + u^2) \]

with 2 sample points (blue Lines).
Different $r$ Values

By choosing different $r$ values we are able to get a number of different equilibrium points. Here we have three equilibrium points when $r$ is equal to $0.383971$.

$$r \left(1 - \frac{u}{q}\right) = \frac{u}{1 + u^2}$$
\[ u' = r u \left(1 - \frac{u}{q}\right) - \frac{u^2}{1 + u^2} \]
\[ u' = ru (1 - u/q) - u^2/(1 + u^2) \]
Four Equilibrium Points

\[ r = 0.5 \]

\[ r \left(1 - \frac{u}{q}\right) = \frac{u}{1 + u^2} \]
\[ u' = r \ u \ (1 - u/q) - \frac{u^2}{1 + u^2} \]
\[ u' = r u (1 - u/q) - \frac{u^2}{1 + u^2} \]
Three Equilibrium Points

\[ r = 0.559525 \]

\[ r \left(1 - \frac{u}{q}\right) = \frac{u}{1 + u^2} \]
\[ u' = r \ u \ (1 - u/q) - u^2/(1 + u^2) \]
\[ u' = r u (1 - u/q) - u^2/(1 + u^2) \]
Two Equilibrium Points

\[ r = 0.645 \]

\[ r \left(1 - \frac{u}{q}\right) = \frac{u}{1 + u^2} \]
\[ u' = r \ u \ (1 - u/q) - u^2/(1 + u^2) \]
\[ u' = r u (1 - u/q) - \frac{u^2}{1 + u^2} \]
Cusp
The Hysteresis Effect of the Curves on the ru Plane
Bibliography